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Extremum principles for slow viscous flows with applications to suspensions

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Helmholtz stated and Korteweg proved that of all divergenceless velocity fields in a domain, with prescribed values on the boundary, the solution of the Stokes equation minimizes the rate of viscous energy dissipation. Hill & Power and also Kearsley proved the corresponding reciprocal maximum principle involving the stress tensor. We prove generalizations of both these principles to the flow of a liquid containing one or more solid bodies and drops of another liquid. The essential point in doing this is to take account of the motion of the solids or drops, which must be determined along with the flow. We illustrate the use of these principles by deducing several consequences from them. In particular we obtain upper and lower bounds on the effective coefficient of viscosity and a lower bound on the sedimentation velocity of suspensions of any concentration. The results involve the statistical properties of the distribution of suspended particles or drops. Graphs of the bounds are shown for special cases. For very low concentrations of spheres, both bounds on the effective viscosity coefficient are the same, and agree with the results of Einstein and Taylor.

1. Introduction

Helmholtz (1868) studied the slow steady (Stokes) flow of an incompressible viscous fluid with a given velocity on its boundary and subject to forces derivable from a single valued potential. He proved that the rate of energy dissipation for such a flow is stationary in a certain class of flows. However he stated that it was a minimum, which Korteweg (1883) then proved. Rayleigh (1913) obtained a similar result for non-slow flows with harmonic vorticity. Much later Hill & Power (1956) and then Kearsley (1960) proved a maximum principle complementary to the minimum principle of Helmholtz and Korteweg. We shall generalize these principles to Stokes flows containing one or more solid bodies, drops of another liquid, or gas bubbles whose motions are unspecified, and which must therefore be determined along with the flow. Then we shall apply these new principles to obtain bounds on the effective viscosity and sedimentation velocity of a suspension of any concentration.

First we shall rederive the previous extremum principles under more general boundary conditions than has been done before, a possibility that Helmholtz and Hill & Power indicated. We shall also relax the smoothness requirements on the comparison flows. This is of some practical value because it makes it easier to construct comparison flows. In addition, we shall admit forces not derivable from a potential, as Helmholtz did, and obtain modified extremum principles. They reduce to the previous ones when the forces are derivable from a potential. Then we shall formulate the problem of the Stokes flow of a fluid containing moving solid, liquid or gaseous objects whose motions must be determined simultaneously with the flow. For this problem we shall then prove the generalized extremum principles.

From these principles we shall prove the uniqueness of the Stokes flow of a fluid containing moving objects, as Helmholtz and Korteweg did in the special case they considered. We shall also show that the exact steady or unsteady Navier–Stokes flow yields a larger drag and torque on a body than does the Stokes flow. Many other applications of these principles are possible, and some of them have been indicated by Hill & Power. We shall also explain the bearing of the minimum principle on the principle of the minimum rate of entropy production, which is used in irreversible thermodynamics.

An account of most of the previous work on Stokes flows, with applications to suspensions, is contained in the recent book of Happel & Brenner (1965).

Hashin (1962) and Prager (1963) have tried to obtain bounds on the viscosity of a suspension, Hashin by using the Helmholtz minimum principle and Prager by using the principle of the minimum rate of entropy production, which is equivalent to the Helmholtz principle for viscous flow. Hashin (1967) also tried to obtain bounds on the viscosity of a mixture by using other variational principles. None of these principles takes account of the fact that the motion of the suspended particles or droplets is unknown. Valid bounds cannot be obtained in this way unless the particle motions are prescribed.

2. A minimum principle for the Stokes flow

Let us consider the flow of a viscous incompressible fluid in a domain V bounded by a surface S which is divided into three parts, S_1, S_2 and S_3. We assume that S is smooth enough for Gauss' theorem to be applicable. Let $u(x)$, $p(x)$, μ and $f(x)$ denote respectively the fluid velocity, pressure, viscosity coefficient and external force per unit volume. In terms of these quantities we define the strain rate tensor $e_{ij}[u]$ and the stress tensor $\tau_{ij}[u]$ by

$$e_{ij}[u] = \tfrac{1}{2}(u_{i,j}+u_{j,i}), \tag{2.1}$$

$$\tau_{ij}[u] = 2\mu e_{ij}[u]-p\,\delta_{ij}. \tag{2.2}$$

Here and elsewhere u_i denotes the ith Cartesian component of u, $u_{i,j}$ denotes its derivative with respect to the Cartesian co-ordinate x_j and a term bearing a repeated index is to be summed over the values 1, 2 and 3 of that index.

We wish to determine u and p in V satisfying the following conditions:

$$u_{i,i} = 0, \qquad x \text{ in } V; \tag{2.3}$$

$$\tau_{ij,j}+f_i = 0, \qquad x \text{ in } V; \tag{2.4}$$

$$u_i(x) = g_i(x), \qquad x \text{ on } S_1; \tag{2.5}$$

$$n_i u_i(x) = h(x), \qquad x \text{ on } S_2; \tag{2.6}$$

$$n_j \tau_{ij} - n_i n_k n_m \tau_{km} = \beta_i, n_i \beta_i = 0, \quad x \text{ on } S_2; \tag{2.7}$$

$$n_j \tau_{ij} = \gamma_i, \qquad x \text{ on } S_3. \tag{2.8}$$

In (2.6)–(2.8) n_i denotes the ith component of the unit normal to S pointing out of V. Equation (2.3) expresses the incompressibility of the fluid and (2.4) is the Stokes equation, which results from the Navier–Stokes equation when the inertia terms $u_t + (u.\nabla)u$ are omitted. These terms can be validly omitted when the flow is so slow and changes so slowly that viscous forces are more important than inertial forces. Equation (2.5) specifies that the ith component of the velocity must be $g_i(x)$ on S_1, (2.6) specifies that the normal component of velocity must be $h(x)$ on S_2, (2.7) specifies that the tangential component of normal stress equals β_i on S_2 and (2.8) specifies that the normal stress on S_3 equals γ_i. We may think of S_1 as the surface of contact between the fluid and a solid while S_2 may be a surface of contact with a gas. A solution $u(x)$ of (2.3)–(2.8) will be called a Stokes flow.

The rate per unit volume at which mechanical energy is converted into heat is $\tau_{ij}[u]\, e_{ij}[u]$. The volume integral of this quantity is defined to be the dissipation rate $D[u]$. By virtue of (2.3) this is

$$D[u] = 2\int_V \mu(e_{ij}[u])^2\, dV. \tag{2.9}$$

The excess dissipation rate $D_e[u]$ is defined to be the rate of energy dissipation minus twice the power of the external body forces and given surface tractions. Upon using (2.7) and (2.8) we can write

$$D_e[u] = D[u] - 2\int_V f_i u_i\, dV - 2\int_{S_2} \beta_i u_i\, dS - 2\int_{S_3} \gamma_i u_i\, dS. \tag{2.10}$$

We shall consider flows for which all the integrals that occur exist. For domains which extend to infinity we also require the vanishing of those surface integrals over the sphere at infinity which arise in using Gauss' theorem. We shall not mention these conditions explicitly again.

Our aim is to show that the Stokes flow minimizes $D_e[u]$ in a certain class of functions. To do so we need the following lemma:

Lemma 1. Let V be a domain bounded by a surface S within which is defined a continuously differentiable flow $u(x)$ satisfying (2.3) and (2.4). Let $v(x)$ be a continuous flow defined in V, possessing piecewise continuous derivatives, and satisfying (2.3). Then

$$D_e[u+v] = D_e[u] + D[v] + 2\int_S v_i \tau_{ij}[u]\, n_j\, dS - 2\int_{S_2} \beta_i v_i\, dS - 2\int_{S_3} \gamma_i v_i\, dS. \tag{2.11}$$

Proof. From (2.1) we have

$$e^2_{ij}[u+v] = e^2_{ij}[u] + e^2_{ij}[v] + 2e_{ij}[u]\, e_{ij}[v]. \tag{2.12}$$

Now from (2.9) and (2.12) we obtain

$$D[u+v] = D[u] + D[v] + \int_V 4\mu e_{ij}[u]\, e_{ij}[v]\, dV. \tag{2.13}$$

By noting that $e_{ij} = e_{ji}$ and using $v_{i,i} = 0$ and (2.4) we also have the relation

$$\begin{aligned} 4\mu e_{ij}[v]\, e_{ij}[u] &= 2\mu(v_{i,j} + v_{j,i})\, e_{ij}[u] \\ &= 4\mu e_{ij}[u]\, v_{i,j} \\ &= 2\tau_{ij}[u]\, v_{i,j} \\ &= 2\partial_j(v_i \tau_{ij}[u]) + 2f_i v_i. \end{aligned} \tag{2.14}$$

Using (2.13) and (2.14) in (2.10) we find

$$D_e[u+v] = D_e[u] + D[v] + 2\int_V \partial_j(v_i \tau_{ij}[u])\, dS - 2\int_{S_2} \beta_i v_i\, dS - 2\int_{S_3} \gamma_i v_i\, dS. \tag{2.15}$$

We now use Gauss' theorem to convert the first integral in (2.15) to the first surface integral in (2.11). In doing so we must take account of the surfaces across which the derivatives of v are discontinuous. Since v is continuous across these surfaces, the surface integrals over the two sides of each such surface cancel. This proves the lemma.

We can now state and prove the following theorem:

Theorem 1. (*Minimum Principle.*) Let $u(x)$ be a continuously differentiable solution of (2.3)–(2.8). Let $w(x)$ be any continuous and piecewise continuously differentiable flow defined in V satisfying (2.3), (2.5) and (2.6). Then

$$D_e[w] \geqslant D_e[u]. \tag{2.16}$$

The inequality holds in (2.16) if $w(x) - u(x)$ is not a rigid body motion. Thus inequality holds if $w(x) \neq u(x)$ provided S_1 is not empty or provided no rigid body motion of the fluid is possible in V with vanishing normal velocity on S_2.

Proof. If we set $w = u+v$ then v satisfies (2.3) and vanishes on S_1 while $n_i v_i$ vanishes on S_2. Therefore (2.11) applies. Now in (2.11) the surface integral over S_1 vanishes because $v_i = 0$ there. On S_2 the integrand of the first surface integral becomes $v_i(\beta_i + n_i n_k n_m \tau_{km})$, and this reduces to $v_i \beta_i$ since $n_i v_i = 0$ on S_2. Twice the integral of this over S_2 cancels the fourth term in (2.11). The remaining integrals in (2.11) also cancel because of (2.8). Thus (2.11) becomes

$$D_e[u+v] = D_e[u] + D[v]. \tag{2.17}$$

Since $D[v] \geqslant 0$, this yields (2.16). Equality holds in (2.17) if and only if $D[v] = 0$, which is so only if v is a rigid body motion. Then since $v = 0$ on S_1, it follows that $v \equiv 0$ provided S_1 is not empty. When S_1 is empty then $v \equiv 0$ if no rigid motion is possible in V with vanishing normal velocity on S_2. This completes the proof of the theorem.

For the case in which S_2 and S_3 are absent, Helmholtz showed that D_e is stationary for the Stokes flow and Korteweg proved that it is a minimum.

3. A maximum principle for the Stokes flows

Theorem 1 enables us to get upper bounds on $D_e[u]$ for a Stokes flow u. We shall now show how to get lower bounds on $D_e[u]$ by deriving a maximum principle for the Stokes flows. This principle is related to the minimum principle by means of the Friedrichs (1929) transformation (Courant & Hilbert 1953). It involves a functional $H[\sigma_{ij}]$ of a tensor σ_{ij}, defined by

$$H[\sigma_{ij}] = -\frac{1}{2\mu}\int_V (\sigma_{ij}-\tfrac{1}{3}\sigma_{kk}\delta_{ij})^2\,dV + 2\int_{S_1} g_i n_j \sigma_{ij}\,dS + 2\int_{S_2} h n_i n_j \sigma_{ij}\,dS. \tag{3.1}$$

We shall first prove that H is related to D_e as follows:

$$H[\tau_{ij}[u]] = D_e[u]. \tag{3.2}$$

This equality states that when τ_{ij} is the stress tensor (2.2) corresponding to a Stokes flow u satisfying (2.3)–(2.8) then $H[\tau_{ij}]$ is equal to $D_e[u]$.

To prove (3.2) we use (2.5) and (2.6) to write the integrands of the surface integrals in (3.1) as $u_i n_j \tau_{ij}$ and $u_k n_k n_i n_j \tau_{ij}$ respectively. By (2.7)

$$u_k n_k n_i n_j \tau_{ij} = u_k n_r \tau_{kr} - \beta_k u_k.$$

We now add and subtract to the right side of (3.1) the integral of $u_k n_r \tau_{kr}$ over S_3. Then we apply Gauss' theorem to the integrals of $u_k n_r \tau_{kr}$ in (3.1) and obtain

$$H[\tau_{ij}[u]] = -\frac{1}{2\mu}\int_V (\tau_{ij}-\tfrac{1}{3}\tau_{kk}\delta_{ij})^2\,dV + 2\int_V \partial_j(u_i\tau_{ij})\,dV - 2\int_{S_2}\beta_i u_i\,dS - 2\int_{S_3} u_i n_j \tau_{ij}\,dS. \tag{3.3}$$

Upon differentiating the second integrand it becomes $u_{i,j}\tau_{ij} + u_i\tau_{ij,j}$. The first of these terms is equal to $2\mu(e_{ij})^2$ as we see from (2.1) to (2.3). The second term is $-u_i f_i$, according to (2.4). The first integrand in (3.3) is equal to $(2\mu e_{ij})^2$ while the last integrand is $u_i\gamma_i$ by (2.8). Thus (3.3) becomes

$$H[\tau_{ij}[u]] = -2\mu\int_V (e_{ij})^2\,dV + 4\mu\int_V (e_{ij})^2\,dV - 2\int_V u_i f_i\,dV - 2\int_{S_2}\beta_i u_i\,dS - 2\int_{S_3}\gamma_i u_i\,dS. \tag{3.4}$$

From the definition (2.10) we see that the right side of (3.4) is $D_e[u]$, which proves (3.2).

In order to prove the maximum principle we shall need the following lemma:

Lemma 2. Let V be a domain bounded by a surface S within which is defined a continuously differentiable flow $u(x)$, satisfying (2.3), in terms of which $\tau_{ij}[u]$ is given by (2.2). Let $\sigma_{ij}(x)$ be a piecewise continuous and piecewise continuously differentiable tensor defined in V and satisfying the conditions

$$\sigma_{ij} = \sigma_{ji}, \tag{3.5}$$

$$n'_i\sigma_{ij} \text{ continuous across the discontinuity surfaces of } \sigma_{ij}. \tag{3.6}$$

Then

$$H[\tau_{ij}[u]+\sigma_{ij}] = H[\tau_{ij}[u]] - \frac{1}{2\mu}\int_V (\sigma_{ij}-\tfrac{1}{3}\sigma_{kk}\delta_{ij})^2\,dV + 2\int_V u_i\sigma_{ij,j}\,dV + 2\int_{S_1}(g_i-u_i)n_j\sigma_{ij}\,dS + 2\int_{S_2}(hn_i-u_i)n_j\sigma_{ij}\,dS - 2\int_{S_3}u_i n_j\sigma_{ij}\,dS. \tag{3.7}$$

In (3.6) n'_i denotes the unit normal to a discontinuity surface of σ_{ij}.

Proof. From (3.1) we have

$$H[\tau_{ij}[u]+\sigma_{ij}] = H[\tau_{ij}[u]]+H[\sigma_{ij}]-\frac{1}{\mu}\int_V (\tau_{ij}[u]-\tfrac{1}{3}\tau_{kk}[u]\,\delta_{ij})(\sigma_{ij}-\tfrac{1}{3}\sigma_{mm}\,\delta_{ij})\,dV. \tag{3.8}$$

By using (2.1)–(2.3) and (3.5) we can rewrite the integrand in the last integral as follows:

$$\begin{aligned}(\tau_{ij}[u]-\tfrac{1}{3}\tau_{kk}[u]\,\delta_{ij})(\sigma_{ij}-\tfrac{1}{3}\sigma_{mm}\,\delta_{ij}) &= 2\mu e_{ij}[u]\,(\sigma_{ij}-\tfrac{1}{3}\sigma_{mm}\,\delta_{ij})\\ &= \mu(u_{i,j}+u_{j,i})\,\sigma_{ij}\\ &= 2\mu u_{i,j}\,\sigma_{ij}\\ &= 2\mu\,\partial_j(u_i\,\sigma_{ij})-2\mu u_i\,\partial_j\,\sigma_{ij}.\end{aligned} \tag{3.9}$$

Upon substituting the last expression into (3.8) and using Gauss' theorem to convert the integral of $\partial_j(u_i\,\sigma_{ij})$ to a surface integral, we obtain (3.7). There are other surface integrals over the surfaces of discontinuity of σ_{ij}, but because of (3.6) the integrals over the two sides of each of these surfaces cancel. This proves the lemma.

We can now state and prove the maximum principle.

Theorem 2. (*Maximum Principle.*) Let $u(x)$ be a continuously differentiable solution of (2.3)–(2.8). Let $\sigma_{ij}(x)$ be any piecewise continuous and piecewise continuously differentiable tensor defined in V satisfying (3.5), (3.6), (2.4), (2.7) and (2.8). Then

$$H[\sigma_{ij}] \leqslant H[\tau_{ij}[u]] = D_e[u]. \tag{3.10}$$

Inequality holds in (3.10) unless $\sigma_{ij} = \tau_{ij}[u]+q_0\,\delta_{ij}$, where q_0 is a constant which is zero unless S_3 is absent.

Proof. Let us set $\sigma_{ij} = \tau_{ij}[u]+\rho_{ij}$. Then ρ_{ij} is piecewise continuous and piecewise continuously differentiable in V and satisfies (3.5), (3.6), (2.7) and (2.8) with $\beta_i = \gamma_i = 0$, and (2.4) with $f_i = 0$, i.e. $\rho_{ij,j} = 0$. Therefore (3.7) holds with ρ_{ij} instead of σ_{ij}. From (2.5) the first surface integral in (3.7) vanishes and, because $\rho_{ij,j} = 0$, the second volume integral vanishes. The second surface integral vanishes in virtue of (2.6) and (2.7) with $\beta_i = 0$, since

$$(hn_i-u_i)\,n_j\,\rho_{ij} = (hn_i-u_i)\,n_i\,n_k\,n_m\,\rho_{km} = (hn_i^2-u_i\,n_i)\,n_k\,n_m\,\rho_{km} = 0.$$

The last integral vanishes by (2.8) with $\gamma_i = 0$. Thus (3.7) becomes

$$H[\tau_{ij}[u]+\rho_{ij}] = H[\tau_{ij}[u]]-\frac{1}{2\mu}\int_V (\rho_{ij}-\tfrac{1}{3}\rho_{kk}\,\delta_{ij})^2\,dV. \tag{3.11}$$

The left side of (3.11) is $H[\sigma_{ij}]$, the integral term is negative or zero and

$$H[\tau_{ij}[u]] = D_e[u]$$

according to (3.2). Thus (3.11) implies (3.10). The proof of the theorem is completed by noting that the integral in (3.11) vanishes if and only if $\rho_{ij} = q(x)\,\delta_{ij}$ and, since $\rho_{ij,j} = 0$, it follows that $q(x) = q_0 =$ constant. Furthermore, since $n_i\,\rho_{ij} = 0$ on S_3, we have $q_0 = 0$ unless S_3 is absent.

Theorems 1 and 2 yield both upper and lower bounds on $D_e[u]$,

$$H[\sigma_{ij}] \leqslant D_e[u] \leqslant D_e[w]. \tag{3.12}$$

Before considering the applications of these bounds, we shall obtain similar bounds for flows containing moving rigid bodies, droplets or gas bubbles. A special case of theorem 2 in which S_2 and S_3 are absent, $g_i = 0$ and $f_i = 0$, was proved by Hill & Power.

4. A minimum principle for slow flows containing solid or fluid particles

We intend to apply our extremum principles to flows of suspensions, which are fluids containing solid particles or drops of a different fluid. In such flows the velocities of the particles or drops must be determined simultaneously with the flow. Thus the velocity of the fluid at the surface of a particle or drop is not known in advance. As a consequence our previous theorems are not applicable to such flows. We shall now derive new extremum principles which are applicable.

Let us begin by formulating the problem of the slow motion of an incompressible viscous fluid containing N fluid drops with surfaces $s_1, \ldots, s_N$ and M rigid particles with surfaces $s_{N+1}, \ldots, s_{N+M}$. Then the domain V is bounded by these $N+M$ surfaces and by the surfaces S_1, S_2 and S_3. We shall suppose that the motion of each surface s_k is a rigid body motion characterized by the velocity $U^{(k)}$ of some reference point inside the surface and an angular velocity $\omega^{(k)}$ about an axis through this point. This means that change of shape of the fluid drops is not considered, which is valid when the surface tension is large enough. Let $u^{(k)}(x)$, $p^{(k)}(x)$, $\tau^{(k)}_{ij}(x)$, $\mu^{(k)}$ and $f^{(k)}(x)$ denote respectively the fluid velocity, pressure, stress tensor, viscosity coefficient and external force per unit volume of the kth fluid drop in the domain V_k bounded by s_k, $k = 1, \ldots, N$. Let $n^{(k)}$ denote the unit normal to s_k pointing out of particle k and let $F^{(k)}$ and $N^{(k)}$ denote respectively the external force and torque on particle k. Finally let $r^{(k)}$ denote the position vector from the reference point in particle k.

Now we can formulate the flow problem. It is to find $u(x)$ defined in V, $u^{(k)}(x)$ defined in V_k, $k = 1, \ldots, N$, $U^{(k)}$ and $\omega^{(k)}$, $k = 1, \ldots, N+M$, subject to the following conditions:

$$u(x) \text{ satisfies (2.3)–(2.8),} \tag{4.0}$$

$$u_i = U^{(k)}_i + \epsilon_{imj}\,\omega^{(k)}_m r^{(k)}_j, \quad x \text{ on } s_k \quad (k = N+1, \ldots, N+M). \tag{4.1}$$

$$u_i = u^{(k)}_i, \quad x \text{ on } s_k \quad (k = 1, \ldots, N), \tag{4.2}$$

$$n^{(k)}_i u_i = n^{(k)}_i u^{(k)}_i = n^{(k)}_i\,[U^{(k)}_i + \epsilon_{imj}\,\omega^{(k)}_m r^{(k)}_j], \quad x \text{ on } s_k \quad (k = 1, \ldots, N), \tag{4.3}$$

$$n^{(k)}_j(\tau_{ij} - \tau^{(k)}_{ij}) = n^{(k)}_i n^{(k)}_q n^{(k)}_m(\tau_{qm} - \tau^{(k)}_{qm}), \quad x \text{ on } s_k \quad (k = 1, \ldots, N), \tag{4.4}$$

$$u^{(k)}_{i,i}(x) = 0, \quad x \text{ in } V_k \quad (k = 1, \ldots, N), \tag{4.5}$$

$$\tau^{(k)}_{ij,j} + f^{(k)}_i = 0, \quad x \text{ in } V_k \quad (k = 1, \ldots, N), \tag{4.6}$$

$$\int_{V_k} f^{(k)}_i\,dV + \int_{s_k} n^{(k)}_j \tau_{ij}[u]\,ds = 0 \quad (k = 1, \ldots, N), \tag{4.7}$$

$$F^{(k)}_i + \int_{s_k} n^{(k)}_j \tau_{ij}[u]\,ds = 0 \quad (k = N+1, \ldots, N+M), \tag{4.8}$$

$$\int_{V_k} \epsilon_{iqj}\, r^{(k)}_q f^{(k)}_j\,dV + \int_{s_k} \epsilon_{iqj}\, r^{(k)}_q n^{(k)}_m \tau_{jm}[u]\,ds = 0 \quad (k = 1, \ldots, N), \tag{4.9}$$

$$N^{(k)}_i + \int_{s_k} \epsilon_{iqj}\, r^{(k)}_q n^{(k)}_m \tau_{jm}[u]\,ds = 0 \quad (k = N+1, \ldots, N+M). \tag{4.10}$$

In these equations $\epsilon_{iqj} = +1$ or -1 according as iqj is an even or odd permutation of 1, 2, 3 and $\epsilon_{iqj} = 0$ otherwise.

Equation (4.1), which equates the fluid velocity at the surface of a rigid particle to that of the surface, is the no slip condition. Equation (4.2) equates the velocities of the fluid on the two sides of the surface of a drop and (4.3) equates their normal components to the normal velocity of the surface. Equation (4.4) expresses the continuity of the tangential component of normal stress across the surface of each drop. Equation (4.5) expresses incompressibility of the fluid in each drop and (4.6) is the Stokes equation of motion for the fluid. Equations (4.7)–(4.10) are the equations of motion of the drops and particles when the inertial terms are negligible.

We now define the excess dissipation rate $D_e[u, u^{(k)}, U^{(k)}, \omega^{(k)}]$ by

$$D_e[u, u^{(k)}, U^{(k)}, \omega^{(k)}] = D_e[u] + \sum_{k=1}^{N} D^{(k)}[u^{(k)}] - 2\sum_{k=1}^{N}\int_{V_k} f_i^{(k)} u_i^{(k)}\, dV - 2\sum_{k=N+1}^{N+M}[F_i^{(k)} U_i^{(k)} + N_i^{(k)}\omega_i^{(k)}]. \quad (4.11)$$

Here $D^{(k)}[u^{(k)}]$ is the dissipation rate in V_k $(k = 1, \ldots, N)$. We can now state and prove the following minimum principle:

Theorem 3. (*Minimum Principle.*) Let

$$u(x), \quad u^{(k)}(x) \quad (k = 1, \ldots, N), \qquad U^{(k)}, \quad \omega^{(k)} \quad (k = 1, \ldots, N+M)$$

be a solution of (4.0)–(4.10) with $u(x)$ and $u^{(k)}(x)$ continuously differentiable in V and V_k respectively. Let $\bar{u}(x)$, $\bar{u}^{(k)}(x)$ $(k = 1, \ldots, N)$, $\bar{U}^{(k)}, \bar{\omega}^{(k)}$ $(k = 1, \ldots, N+M)$ satisfy (2.3), (2.5), (2.6), (4.1), (4.2), (4.3) and (4.5) with $\bar{u}(x)$ and $\bar{u}^{(k)}(x)$ being continuous and piecewise continuously differentiable in V and V_k respectively. Then

$$D_e[\bar{u}, \bar{u}^{(k)}, \bar{U}^{(k)}, \bar{\omega}^{(k)}] \geqslant D_e[u, u^{(k)}, U^{(k)}, \omega^{(k)}]. \quad (4.12)$$

The inequality holds in (4.12) when $\bar{u} - u$, $\bar{u}^{(k)} - u^{(k)}$, $\bar{U}^{(k)} - U^{(k)}$ and $\bar{\omega}^{(k)} - \omega^{(k)}$ is not a rigid body motion. Thus inequality holds when

$$[\bar{u}(x), \bar{u}^{(k)}(x), \bar{U}^{(k)}, \bar{\omega}^{(k)}] \not\equiv [u(x), u^{(k)}(x), U^{(k)}, \omega^{(k)}]$$

if S_1 is not empty or if no rigid body motion of the fluid is possible in V with vanishing normal derivative on the surface S_2.

Proof. Let us write

$$\bar{u}(x) = u(x) + \tilde{u}(x), \quad \bar{u}^{(k)}(x) = u^{(k)}(x) + \tilde{u}^{(k)}(x) \quad (k = 1, \ldots, N),$$
$$\bar{U}^{(k)} = U^{(k)} + \tilde{U}^{(k)}, \qquad \bar{\omega}^{(k)} = \omega^{(k)} + \tilde{\omega}^{(k)} \quad (k = 1, \ldots, N+M).$$

Then $\tilde{u}$ and $\tilde{u}^k$ are continuous and piecewise continuously differentiable in V and V_k respectively for $k = 1, \ldots, N$; $\tilde{u}_i = 0$ on S_1 and $n_i \tilde{u}_i = 0$ on S_2. In addition $\tilde{u}, \tilde{u}^{(k)}, \tilde{U}^{(k)}$ and $\tilde{\omega}^{(k)}$ satisfy (2.3), (4.1)–(4.3) and (4.5). Upon substituting $\bar{u} = u + \tilde{u}$, etc., into (4.11) we obtain

$$D_e[\bar{u}, \bar{u}^{(k)}, \bar{U}^{(k)}, \bar{\omega}^{(k)}] = D_e[u + \tilde{u}] + \sum_{k=1}^{N} D^{(k)}[u^{(k)} + \tilde{u}^{(k)}] - 2\sum_{k=N+1}^{N+M}[F_i^{(k)} U_i^{(k)} + N_i^{(k)}\omega_i^{(k)}] - 2\sum_{k=N+1}^{N+M}[F_i^{(k)} \tilde{U}_i^{(k)} + N_i^{(k)}\tilde{\omega}_i^{(k)}] - 2\sum_{k=1}^{N}\int_{V_k} f_i^{(k)}\,(u^{(k)} + \tilde{u}_i^{(k)})\, dV. \quad (4.13)$$

Now we use Lemma 1, which is applicable to u and $\tilde{u}$ in V, and equations (2.13) and (2.14), which are applicable to $u^{(k)}$ and $\tilde{u}^{(k)}$ in V_k. Then (4.13) becomes

$$\begin{aligned} D_e[\bar{u}, \bar{u}^{(k)}, \bar{U}^{(k)}, \bar{\omega}^{(k)}] &= D_e[u] + \sum_{k=1}^{N} D^{(k)}[u^{(k)}] - 2\sum_{k=1}^{N}\int_{V_k} f_i^{(k)} u_i^{(k)}\, dV \\ &\quad - 2\sum_{k=N+1}^{N+M} [f_i^{(k)} U_i^{(k)} + N_i^{(k)}\omega_i^{(k)}] + D[\tilde{u}] + \sum_{k=1}^{N} D^{(k)}[\tilde{u}^{(k)}] \\ &\quad - 2\sum_{k=N+1}^{N+M} [F_i^{(k)} \tilde{U}_i^{(k)} + N_i^{(k)}\tilde{\omega}_i^{(k)}] - 2\sum_{k=1}^{N+M}\int_{s_k} \tilde{u}_i \tau_{ij}[u]\, n_j^{(k)}\, dS \\ &\quad + 2\sum_{k=1}^{N}\int_{V_k} \partial_j(\tilde{u}_i^{(k)}\tau_{ij}[u^{(k)}])\, dV + 2\int_{S_1+S_2+S_3} \tilde{u}_i\tau_{ij}[u]\, n_j\, dS \\ &\quad - 2\int_{S_2} \beta_i \tilde{u}_i\, dS - 2\int_{S_3} \gamma_i \tilde{u}_i\, dS. \end{aligned} \tag{4.14}$$

We have used the fact that the normal to s_k out of V is just $-n^{(k)}$.

The first term plus the next three sums are just equal to $D_e[u, u^{(k)}, U^{(k)}, \omega^{(k)}]$ as we see from (4.11). The integrals over S_1, S_2 and S_3 add up to zero by virtue of the boundary conditions satisfied by $\tilde{u}_i$ and $\tau_{ij}[u]$, as is shown in the proof of theorem 1. The integrals over s_k, $k = N+1, \ldots, N+M$, can be rewritten as follows by using (4.1),

$$\begin{aligned} &-2\sum_{k=N+1}^{N+M}\int_{s_k} \tilde{u}_i\tau_{ij}[u]\, n_j^{(k)}\, dS \\ &\quad = -2\sum_{k=N+1}^{N+M}\left[\tilde{U}_i^{(k)}\int_{s_k}\tau_{ij}[u]\, n_j^{(k)} dS + \omega_m^{(k)}\int_{s_k}\epsilon_{imj}\, r_j^{(k)}\tau_{iq}[u]\, n_q^{(k)}\, dS\right]. \end{aligned} \tag{4.15}$$

Now, we add $-2\sum_{k=N+1}^{M+N}[F_i^{(k)}\tilde{U}_i^{(k)} + N_i^{(k)}\tilde{\omega}_i^{(k)}]$ to both sides of (4.15). Then as a consequence of (4.8) and (4.10) the right side of the resulting equation is zero. The integral over V_k $(k = 1, \ldots, N)$ can be converted, with the aid of Gauss' theorem, to an integral over s_k. By combining this with the remaining integrals over s_k $(k = 1, \ldots, N)$ we can write (4.14) as

$$\begin{aligned} D_e[\bar{u}, \bar{u}^{(k)}, \bar{U}^{(k)}, \bar{\omega}^{(k)}] &= D_e[u, u^{(k)}, U^{(k)}, \omega^{(k)}] + D[\tilde{u}] + \sum_{k=1}^{N} D^{(k)}[\tilde{u}^{(k)}] \\ &\quad - 2\sum_{k=1}^{N}\int_{s_k} (\tilde{u}_i\tau_{ij}[u] - \tilde{u}_i^{(k)}\tau_{ij}[u^{(k)}])\, n_j^{(k)}\, dS. \end{aligned} \tag{4.16}$$

Let us consider the integrand of the integral over s_k. By successively using (4.2), (4.4), (4.3) and (4.4) we can write

$$\begin{aligned} (\tilde{u}_i\tau_{ij}[u] - \tilde{u}_i^{(k)}\tau_{ij}[u^{(k)}])\, n_j^{(k)} &= \tilde{u}_i(\tau_{ij}[u] - \tau_{ij}[u^{(k)}])\, n_j^{(k)} \\ &= \tilde{u}_i\, n_i^{(k)} n_q^{(k)} n_m^{(k)} (\tau_{qm}[u] - \tau_{qm}[u^{(k)}]) \\ &= (\tilde{U}_i^{(k)} + \epsilon_{imj}\,\omega_m^{(k)} r_j^{(k)})\, n_i^{(k)} n_q^{(k)} n_m^{(k)} (\tau_{qm}[u] - \tau_{qm}[u^{(k)}]) \\ &= (\tilde{U}_i^{(k)} + \epsilon_{imj}\,\omega_m^{(k)} r_j^{(k)})\, n_t^{(k)} (\tau_{ti}[u] - \tau_{ti}[u^{(k)}]). \end{aligned} \tag{4.17}$$

Now the integral over s_k can be written as follows and the second integral converted to a volume integral by Gauss' theorem

$$\int_{s_k} (\tilde{u}_i \tau_{ij}[u] - \tilde{u}_i^{(k)} \tau_{ij}[u^{(k)}])\, n_j^{(k)}\, dS$$
$$= \int_{s_k} (\tilde{U}_i^{(k)} + \epsilon_{imj}\, \omega_m^{(k)} r_j^{(k)})\, \tau_{li}[u]\, n_l^{(k)}\, dS - \int_{s_k} (\tilde{U}_i^{(k)} + \epsilon_{imj}\, \omega_m^{(k)} r_j^{(k)})\, \tau_{li}[u^{(k)}]\, n_l^{(k)}\, dS$$
$$= \int_{s_k} (\tilde{U}_i^{(k)} + \epsilon_{imj}\, \omega_m^{(k)} r_j^{(k)})\, \tau_{li}[u]\, n_l^{(k)}\, dS - \int_{V_k} (U_i^{(k)} + \epsilon_{imj}\, \omega_m^{(k)} r_j^{(k)})\, \tau_{li,\,l}[u^{(k)}]\, dV$$
$$- \int_{V_k} \epsilon_{imj}\, \omega_m^{(k)} r_{j,\,l}^{(k)}\, \tau_{li}[u^{(k)}]\, dV. \qquad (4.18)$$

The last integral in (4.18) vanishes since $r_{j,l}^{(k)} = \delta_{jl}$ and $\epsilon_{imj}\tau_{ji} = 0$ because τ_{ji} is symmetric while ϵ_{imj} is antisymmetric in i and j. We use (4.6) to replace $\tau_{li,\,l}[u^{(k)}]$ by $-f_i^{(k)}$ in the next to last integral in (4.18) and we find that the right side of (4.18) vanishes as a consequence of (4.7) and (4.9). Thus all the integrals in (4.16) vanish and (4.16) becomes

$$D_e[\bar{u}, \bar{u}^{(k)}, \bar{U}^{(k)}, \bar{\omega}^{(k)}] = D_e[u, u^{(k)}, U^{(k)} \omega^{(k)}] + D[\tilde{u}] + \sum_{k=1}^{N} D^{(k)}[\tilde{u}^{(k)}]. \qquad (4.19)$$

Since D and $D^{(k)}$ are non-negative, (4.19) implies (4.12). $D[\tilde{u}]$ and $D^{(k)}[\tilde{u}^{(k)}]$ vanish if and only if $\tilde{u}$ and $\tilde{u}^{(k)}$ represent rigid body motions. By (4.2) they must represent the same motion, and by (2.5) this motion vanishes if S_1 is not empty. If S_1 is empty then this motion vanishes if no rigid body motion is possible in V with vanishing normal velocity on S_2. Thus in these cases the inequality holds in (4.12) provided $u(x) \not\equiv \bar{u}(x)$ and $u^{(k)}(x) \not\equiv \bar{u}^{(k)}(x)$ for $k = 1, \ldots, N$. This completes the proof of the theorem.

This theorem contains theorem 1 as a special case, since when $N = M = 0$, $D_e[u, u^{(k)}, U^{(k)}, \omega^{(k)}] = D_e[u]$. Furthermore, all the conditions in the hypothesis then reduce to those in the hypothesis of theorem 1.

5. A maximum principle for slow flows containing solid or fluid particles

We now define a functional $H[\sigma_{ij}, \sigma_{ij}^{(k)}]$ of a tensor $\sigma_{ij}(x)$ defined in V and N tensors $\sigma_{ij}^{(k)}(x)$ defined in V_k, $k = 1, \ldots N$, by

$$H[\sigma_{ij}, \sigma_{ij}^{(k)}] = -\frac{1}{2\mu}\int_V (\sigma_{ij} - \tfrac{1}{3}\sigma_{nn}\,\delta_{ij})^2\, dV - \sum_{k=1}^{N} \frac{1}{2\mu^{(k)}}\int_{V_k} (\sigma_{ij}^{(k)} - \tfrac{1}{3}\sigma_{nn}^{(k)}\,\delta_{ij})^2\, dV$$
$$+ 2\int_{S_1} g_i\, n_j\, \sigma_{ij}\, dS + 2\int_{S_2} h n_i\, n_j\, \sigma_{ij}\, dS. \qquad (5.1)$$

We shall first show that when $\sigma_{ij} = \tau_{ij}[u]$ and $\sigma_{ij}^{(k)} = \tau_{ij}[u^{(k)}]$ $(k = 1, \ldots, N)$, where u and $u^{(k)}$ are the solutions of (4.0)–(4.10), then

$$H[\tau_{ij}[u], \tau_{ij}[u^{(k)}]] = D_e[u, u^{(k)}, U^{(k)}, \omega^{(k)}]. \qquad (5.2)$$

Thus when σ_{ij} and $\sigma_{ij}^{(k)}$ are the stress tensors of the Stokes flow, H is equal to the excess dissipation rate D_e.

To prove (5.2) we first observe that if $\sigma_{ij} = \tau_{ij}[u]$ the first volume integral term in (5.1) becomes $-D[u]$. Similarly, when $\sigma_{ij}^{(k)} = \tau_{ij}[u^{(k)}]$ the term involving integration over V_k becomes $-D^{(k)}[u^{(k)}]$. Next we use (2.5), according to which $u_i = g_i$ on S_1, to write the integrand in the integral over S_1 as $u_i n_j \tau_{ij}$. Then we use (2.6) and (2.7) to write the integrand in the integral over S_2 as

$$h n_i n_j \tau_{ij} = u_q n_q n_i n_j \tau_{ij} = u_q n_j \tau_{qj} - \beta_q u_q.$$

Then (5.1) becomes

$$H[\tau_{ij}[u], \tau_{ij}[u^{(k)}]] = -D[u] - \sum_{k=1}^{N} D^{(k)}[u^{(k)}] + 2\int_{S_1+S_2} u_j n_i \tau_{ij}[u]\, dS - 2\int_{S_2} \beta_i u_i \, dS. \tag{5.3}$$

We now write the surface integral of $u_j n_i \tau_{ij}[u]$ as an integral over the entire boundary of V minus the integrals over S_3 and the s_k $(k = 1, \ldots, N+M)$. The surface integral over the boundary of V can be converted to a volume integral over V by using Gauss' theorem, and the integrand is $\partial_i(u_j \tau_{ij}) = u_{j,i}\tau_{ij} + u_j \tau_{ij,i}$. From the symmetry of τ_{ij} we have $u_{j,i}\tau_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\tau_{ij} = e_{ij}\tau_{ij} = 2\mu(e_{ij})^2$. The term $u_j \tau_{ij,i}$ can be written as $-u_j f_j$ by noting that $\tau_{ij,i} = -f_j$ according to (2.4). Thus we can rewrite (5.3) as follows

$$\begin{aligned} H[\tau_{ij}[u], \tau_{ij}[u^{(k)}]] &= -D[u] - \sum_{k=1}^{N} D^{(k)}[u^{(k)}] + 2\int_V (2(e_{ij})^2 - u_j f_j)\, dV \\ &\quad + 2\sum_{k=1}^{N+M} \int_{s_k} u_j n_i^{(k)} \tau_{ij}[u]\, dS - 2\int_{S_2} \beta_i u_i \, dS - 2\int_{S_3} u_j n_i \tau_{ij}[u]\, dS. \end{aligned} \tag{5.4}$$

Let us now consider the integral in (5.4) over the surface s_k of a fluid particle so that $1 \leqslant k \leqslant N$. By using successively (4.1), (4.4), (4.3) and (4.4) again we can write the integrand as follows

$$\begin{aligned} u_j^{(k)} n_i^{(k)} \tau_{ij}[u] &= u_j^{(k)} n_i^{(k)} (\tau_{ij}[u] - \tau_{ij}[u^{(k)}]) + u_j^{(k)} n_i^{(k)} \tau_{ij}[u^{(k)}] \\ &= u_j^{(k)}\, n_j^{(k)} n_i^{(k)} n_m^{(k)} (\tau_{im} - \tau_{im}^{(k)}) + u_j^{(k)} n_i^{(k)} \tau_{ij}^{(k)} \\ &= (U_j^{(k)} + \epsilon_{jsq}\, \omega_s^{(k)} r_q^{(k)})\, n_j^{(k)} n_i^{(k)} n_m^{(k)} (\tau_{im} - \tau_{im}^{(k)}) + u_j^{(k)} n_i^{(k)} \tau_{ij}^{(k)}) \\ &= (U_j^{(k)} + \epsilon_{jmq}\, \omega_m^{(k)} r_q^{(k)})\, n_i^{(k)} (\tau_{ij} - \tau_{ij}^{(k)}) + u_j^{(k)} n_i^{(k)} \tau_{ij}^{(k)}. \end{aligned} \tag{5.5}$$

Now we rewrite the surface integral over s_k $(k = 1, \ldots, N)$ using (5.5) and then apply Gauss' theorem to the second integral,

$$\begin{aligned} &\int_{s_k} u_j n_i^{(k)} \tau_{ij}[u]\, dS \\ &= \int_{s_k} (U_j^{(k)} + \epsilon_{jmq}\, \omega_m^{(k)} r_q^{(k)})\, n_i^{(k)} \tau_{ij}[u]\, dS - \int_{s_k} (U_j^{(k)} + \epsilon_{jmq}\, \omega_m^{(k)} r_q^{(k)} - u_j^{(k)})\, n_i^{(k)}\, \tau_{ij}[u^{(k)}]\, dS \\ &= \int_{s_k} (U_j^{(k)} + \epsilon_{jmq}\, \omega_m^{(k)} r_q^{(k)})\, n_i^{(k)} \tau_{ij}[u]\, dS \\ &\qquad - \int_{V_k} \{(U_j^{(k)} + \epsilon_{jmq}\, \omega_m^{(k)} r_q^{(k)} - u_j^{(k)})\, \tau_{ij,i}^{(k)}[u^{(k)}] - u_{j,i}^{(k)}\, \tau_{ij}[u^{(k)}]\}\, dV. \end{aligned} \tag{5.6}$$

Here we have again used the fact that $\epsilon_{jmq}\, r_{q,i}^{(k)} \tau_{ij} = 0$.

The volume integral in (5.6) can be rewritten by setting $\tau_{ij,i}^{(k)} = -f_j^{(k)}$, which follows from (4.6), and observing that $u_{j,i}^{(k)}\tau_{ij}[u^{(k)}] = 2\mu^{(k)}(e_{ij}[u^{(k)}])^2$ for the same reasons as before. Then (5.6) becomes

$$\int_{s_k} u_j n_i^{(k)}\tau_{ij}[u]\,dS = \int_{s_k} (U_j^{(k)} + \epsilon_{jmq}\,\omega_m^{(k)} r_q^{(k)})\, n_i^{(k)}\tau_{ij}[u]\,dS$$
$$+\int_{V_k} \{(U_j^{(k)} + \epsilon_{jmq}\,\omega_m^{(k)} r_q^{(k)} - u_j^{(k)}) f_j^{(k)} + 2\mu^{(k)}(e_{ij}[u^{(k)}])^2\}\,dV. \quad (5.7)$$

In view of (4.7) the integrals in (5.7) proportional to $U_j^{(k)}$ cancel and by (4.9) those proportional to $\omega_m^{(k)}$ cancel. Then when (5.7) is used in (5.4), (5.4) becomes

$$H[\tau_{ij}[u], \tau_{ij}[u^{(k)}]] = -D[u] - \sum_{k=1}^{N} D^{(k)}[u^{(k)}] + 2\int_V 2\mu(e_{ij}[u])^2\,dV - 2\int_V u_i f_i\,dV$$
$$+2\sum_{k=1}^{N}\int_{V_k} 2\mu^{(k)}(e_{ij}[u^{(k)}])^2\,dV - 2\sum_{k=1}^{N}\int_{V_k} u_i^{(k)} f_i^{(k)}\,dV + 2\sum_{k=N+1}^{N+M}\int_{s_k} u_j n_i^{(k)}\tau_{ij}[u]\,dS. \quad (5.8)$$

The first integral over V in (5.8) combines with $-D[u]$ to yield $D[u]$. Similarly, the first integral over V_k combines with $-D^{(k)}[u^{(k)}]$ to yield $D^{(k)}[u^{(k)}]$. The surface integral in (5.8) over s_k $(k = N+1, \ldots, N+M)$ can be simplified by using (4.1). Then (4.8) and (4.10) show that this integral is equal to $-F_i^{(k)} U_i^{(k)} - N_i^{(k)}\omega_i^{(k)}$. Thus (8) becomes

$$H[\tau_{ij}[u], \tau_{ij}[u^{(k)}]] = D[u] + \sum_{k=1}^{N} D^{(k)}[u^{(k)}] - 2\int_V f_i u_i\,dV - 2\sum_{k=1}^{N}\int_{V_k} f_i^{(k)} u_i^{(k)}\,dV$$
$$-2\int_{S_2} \beta_i u_i\,dS - 2\int_{S_1} \gamma_i u_i\,dV - 2\sum_{k=N+1}^{N+M} [F_i^{(k)} U_i^{(k)} + N_i^{(k)}\omega_i^{(k)}]. \quad (5.9)$$

The right side of (5.9) is just $D_e[u, u^{(k)}, U^{(k)}, \omega^{(k)}]$ as we see from (4.11) and (2.10) and this proves (5.2).

In terms of H we now state and prove the following maximum principle.

Theorem 4. (*Maximum Principle.*) Let

$$u(x), \quad u^{(k)}(x) \quad (k = 1, \ldots, N), \quad U^{(k)}, \quad \omega^{(k)} \quad (k = 1, \ldots, N+M)$$

be a solution of (4.0)–(4.10) with $u(x)$ and $u^{(k)}(x)$ continuously differentiable in V and $V^{(k)}$ respectively. Let $\sigma_{ij}(x)$ and $\sigma_{ij}^{(k)}(x)$ be defined in V and V_k respectively $(k = 1, \ldots, N)$ and be piecewise continuous and piecewise continuously differentiable and all satisfy (3.5), (3.6), (4.4), (4.6)–(4.10) and σ_{ij} satisfy (2.4), (2.7) and (2.8). Then

$$H[\sigma_{ij}, \sigma_{ij}^{(k)}] \leqslant H[\tau_{ij}[u], \tau_{ij}[u^{(k)}]] = D_e[u, u^{(k)}, U^{(k)}, \omega^{(k)}]. \quad (5.10)$$

Inequality holds in (5.10) unless $\sigma_{ij} = \tau_{ij}[u] + q_0\delta_{ij}$ and $\sigma_{ij}^{(k)} = \tau_{ij}[u^{(k)}] + q_0^{(k)}\delta_{ij}$, where q_0 and $q_0^{(k)}$ are constants. Furthermore $q_0 = 0$ unless S_3 is absent.

Proof. Let us set $\sigma_{ij} = \tau_{ij}[u] + \rho_{ij}$, $\sigma_{ij}^{(k)} = \tau_{ij}[u^{(k)}] + \rho_{ij}^{(k)}$ $(k = 1, \ldots, N)$. Then ρ_{ij} and $\rho_{ij}^{(k)}$ are piecewise continuous and piecewise continuously differentiable in V and V_k respectively $(k = 1, \ldots, N)$ and they satisfy (3.5), (3.6), (4.4) and (4.6)–

(4.10) with $f_i^{(k)}$, $F_i^{(k)}$ and $N_i^{(k)}$ replaced by zero, and ρ_{ij} satisfies (2.4), (2.7) and (2.8) with f_i, β_i and γ_i replaced by zero. Upon substituting $\sigma_{ij} = \tau_{ij}[u]+\rho_{ij}$ and $\sigma_{ij}^{(k)} = \tau_{ij}[u^{(k)}]+\rho_{ij}^{(k)}$ into (5.1), we get

$$H[\sigma_{ij}, \sigma_{ij}^{(k)}] = H[\tau_{ij}[u]+\rho_{ij}] - \sum_{k=1}^{N} \frac{1}{2\mu^{(k)}} \int_{V_k} (\tau_{ij}[u^{(k)}] - \tfrac{1}{3}\tau_{nn}[u^{(k)}]\,\delta_{ij} + \rho_{ij}^{(k)} - \tfrac{1}{3}\rho_{nn}^{(k)}\delta_{ij})^2\, dV. \quad (5.11)$$

Here $H[\tau_{ij}[u]+\rho_{ij}]$ is defined by (3.1).

We now apply lemma 2 to the first term on the right-hand side of (5.10), taking account of the additional surfaces present, to get

$$\begin{aligned} H[\sigma_{ij}, \sigma_{ij}^{(k)}] = {} & H[\tau_{ij}[u]] - \frac{1}{2\mu}\int_V (\rho_{ij} - \tfrac{1}{3}\rho_{nn}\,\delta_{ij})^2\, dV \\ & + 2\int_V u_i \rho_{ij,j}\, dV + 2\int_{S_1} (g_i - u_i)\, n_j \rho_{ij}\, dS - 2\int_{S_3} u_i n_j \rho_{ij}\, dS \\ & + 2\int_{S_2} (hn_i - u_i)\, n_j \rho_{ij}\, dS + 2\sum_{k=1}^{N+M} \int_{s_k} u_i n_j^{(k)} \rho_{ij}\, dS \\ & - \sum_{k=1}^{N} \frac{1}{2\mu^{(k)}} \int_{V_k} (\tau_{ij}[u^{(k)}] - \tfrac{1}{3}\tau_{nn}[u^{(k)}]\,\delta_{ij} + \rho_{ij}^{(k)} - \tfrac{1}{3}\rho_{nn}^{(k)}\delta_{ij})^2\, dV. \end{aligned} \quad (5.12)$$

The integrals in the last sum in (5.12) are of the same form as $H[\tau_{ij}[u^{(k)}]+\rho_{ij}^{(k)}]$, defined by (3.1), with $g_i = h = 0$ and the surface $S_1+S_2+S_3$ replaced by s_k. Thus we can also apply lemma 2 to these terms to get

$$\begin{aligned} & -\frac{1}{2\mu^{(k)}}\int_{V_k} (\tau_{ij}[u^{(k)}] - \tfrac{1}{3}\tau_{nn}[u^{(k)}]\,\delta_{ij} + \rho_{ij}^{(k)} - \tfrac{1}{3}\rho_{nn}^{(k)}\delta_{ij})^2\, dV \\ & = -\frac{1}{2\mu^{(k)}}\int_{V_k} (\tau_{ij}[u^{(k)}] - \tfrac{1}{3}\tau_{nn}[u^{(k)}]\,\delta_{ij})^2\, dV - \frac{1}{2\mu^{(k)}}\int_{V_k} (\rho_{ij}^{(k)} - \tfrac{1}{3}\rho_{nn}^{(k)}\delta_{ij})^2\, dV \\ & \qquad + 2\int_{V_k} u_i^{(k)} \rho_{ij,j}^{(k)}\, dV - 2\int_{s_k} u_i^{(k)} n_j^{(k)} \rho_{ij}^{(k)}\, dS. \end{aligned} \quad (5.13)$$

Combining (5.13) with (5.12), and rearranging terms, we get

$$\begin{aligned} H[\sigma_{ij}, \sigma_{ij}^{(k)}] = {} & H[\tau_{ij}[u]] - \sum_{k=1}^{N} \frac{1}{2\mu^{(k)}}\int_{V_k} (\tau_{ij}[u^{(k)}] - \tfrac{1}{3}\tau_{nn}[u^{(k)}]\,\delta_{ij})^2\, dV \\ & - \frac{1}{2\mu}\int_V (\rho_{ij} - \tfrac{1}{3}\rho_{nn}\,\delta_{ij})^2\, dV - \sum_{k=1}^{N} \frac{1}{2\mu^{(k)}}\int_{V_k} (\rho_{ij}^{(k)} - \tfrac{1}{3}\rho_{nn}^{(k)}\delta_{ij})^2\, dV \\ & + 2\int_V u_i \rho_{ij,j}\, dV + 2\int_{S_1} (g_i - u_i)\, n_j \rho_{ij}\, dS - 2\int_{S_3} u_i n_j \rho_{ij}\, dS \\ & + 2\int_{S_2} (hn_i - u_i)\, n_j \rho_{ij}\, dS + 2\sum_{k=1}^{N} \int_{V_k} u_i^{(k)} \rho_{ij,j}^{(k)}\, dV \\ & + 2\sum_{k=1}^{N+M} \int_{s_k} u_i n_j^{(k)} \rho_{ij}\, dS - 2\sum_{k=1}^{N} \int_{s_k} u_i^{(k)} n_j^{(k)} \rho_{ij}^{(k)}\, dS. \end{aligned} \quad (5.14)$$

We recognize the first term and the first sum on the right side of (5.14) as $H[\tau_{ij}[u], \tau_{ij}[u^{(k)}]]$. Now, the second integral over V and the third sum in (5.14) vanish because $\rho_{ij,j} = \rho^{(k)}_{ij,j} = 0$. In addition, the integrals over S_1, S_2 and S_3 vanish as was shown in the proof of theorem 2. Therefore (5.14) becomes

$$H[\sigma_{ij}, \sigma^{(k)}_{ij}] = H[\tau_{ij}[u], \tau_{ij}[u^{(k)}]] - \frac{1}{2\mu}\int_V (\rho_{ij} - \tfrac{1}{3}\rho_{nn}\delta_{ij})^2\, dV - \sum_{k=1}^{N} \frac{1}{2\mu^{(k)}}\int_{V_k} (\rho^{(k)}_{ij} - \tfrac{1}{3}\rho^{(k)}_{nn}\delta_{ij})^2\, dV + 2\sum_{k=1}^{N+M}\int_{S_k} u_i n^{(k)}_j \rho_{ij}\, dS - 2\sum_{k=1}^{N}\int_{S_k} u^{(k)}_i n^{(k)}_j \rho^{(k)}_{ij}\, dS. \tag{5.15}$$

By (4.1), we have, for $k = N+1, \ldots, N+M$,

$$\int_{S_k} u_i n^{(k)}_j \rho_{ij}\, dS = \int_{S_k} (U^{(k)}_i + \epsilon_{imq}\omega^{(k)}_m r^{(k)}_q)\, n^{(k)}_j \rho_{ij}\, dS = 0. \tag{5.16}$$

The right side of (5.16) is zero because ρ_{ij} satisfies (4.8) and (4.10) with

$$F^{(k)}_i = N^{(k)}_i = 0.$$

For $k = 1, \ldots, N$ we find by using (4.2), (4.4), (4.3) and (4.4) again that

$$\begin{aligned}\int_{S_k} u_i n^{(k)}_j \rho_{ij}\, dS - \int_{S_k} u^{(k)}_i n^{(k)}_j \rho^{(k)}_{ij}\, dS &= \int_{S_k} u^{(k)}_i n^{(k)}_j (\rho_{ij} - \rho^{(k)}_{ij})\, dS \\ &= \int_{S_k} u^{(k)}_i n^{(k)}_i n^{(k)}_n n^{(k)}_l (\rho_{nl} - \rho^{(k)}_{nl})\, dS \\ &= \int_{S_k} (U^{(k)}_i + \epsilon_{imq}\omega^{(k)}_m r^{(k)}_q)\, n^{(k)}_i n^{(k)}_n n^{(k)}_l (\rho_{nl} - \rho^{(k)}_{nl})\, dS \\ &= \int_{S_k} (U^{(k)}_i + \epsilon_{imq}\omega^{(k)}_m r^{(k)}_q)\, n^{(k)}_j \rho_{ij}\, dS - \int_{S_k} (U^{(k)}_i + \epsilon_{imq}\omega^{(k)}_m r^{(k)}_q)\, n^{(k)}_j \rho^{(k)}_{ij}\, dS.\end{aligned} \tag{5.17}$$

The next to last integral in (5.17) is zero because ρ_{ij} satisfies (4.7) and (4.9) with $f^{(k)}_i = 0$. The last integral in (5.17) can be rewritten, by using Gauss' theorem, as

$$-\int_{S_k} (U^{(k)}_i + \epsilon_{imq}\omega^{(k)}_m r^{(k)}_q)\, n^{(k)}_j \rho^{(k)}_{ij}\, dS = -\int_{V_k} (U^{(k)}_i + \epsilon_{imq}\omega^{(k)}_m r^{(k)}_q)\, \rho^{(k)}_{ij,j}\, dV - \int_{V_k} \epsilon_{imq}\omega^{(k)}_m \rho^{(k)}_{ij} r^{(k)}_{q,j}\, dV. \tag{5.18}$$

The next to last integral in (5.18) vanishes since $\rho^{(k)}_{ij,j} = 0$ in V_k. The last integral likewise vanishes, as was shown in the proof of theorem 2.

By combining all these results we can write (5.15) as

$$H[\sigma_{ij}, \sigma^{(k)}_{ij}] = H[\tau_{ij}[u], \tau_{ij}[u^{(k)}]] - \frac{1}{2\mu}\int_V (\rho_{ij} - \tfrac{1}{3}\rho_{nn}\delta_{ij})^2\, dV - \sum_{k=1}^{N} \frac{1}{2\mu^{(k)}}\int_{V_k} (\rho^{(k)}_{ij} - \tfrac{1}{3}\rho^{(k)}_{nn}\delta_{ij})^2\, dV. \tag{5.19}$$

Since all the integrals in (5.19) are non-negative, (5.19) yields (5.10). Equality holds in (5.10) only if all the integrals in (5.19) vanish and this happens only if $\rho_{ij} = q_0 \delta_{ij}$, $\rho_{ij}^{(k)} = q_0^{(k)}\delta_{ij}$, where q_0 and $q_0^{(k)}$ are constants. Because $n_i \rho_{ij} = 0$ on S_3, $q_0 = 0$ if S_3 is not absent. This proves the theorem.

6. Some consequences of the extremum principles

An immediate consequence of theorem 1 is the following uniqueness theorem.

Theorem 5. (*Uniqueness of the Stokes flow.*) There is in V at most one continuously differentiable Stokes flow, i.e. solution of (2.3)–(2.8), provided S_1 is not empty or provided no rigid body motion of the fluid is possible in V with vanishing normal velocity on S_2.

Proof. If there were two solutions u and w then (2.16) would hold and it would also hold with u and w interchanged, so the equality would hold. But then, by theorem 1, w and u are identical.

In exactly the same way we obtain from theorem 3 the following uniqueness theorem for the Stokes flows containing solid or fluid particles.

Theorem 6. (*Uniqueness of Stokes flow containing particles.*) There is in V and V_k $(k = 1, \ldots, N)$ at most one continuously differentiable Stokes flow and corresponding set of particle velocities and angular velocities, i.e. solution of (4.0)–(4.10), provided S_1 is not empty or provided no rigid body motion of the fluid is possible in V with vanishing normal velocity on the surface S_2.

This theorem contains theorem 5 as a special case.

Suppose f, the external force per unit volume, is derivable from a continuous single valued potential

$$f_i = \Omega,_i. \tag{6.1}$$

Then theorem 1 can be reformulated as follows.

Theorem 7. If $f_i = \Omega,_i$ where Ω is continuous and single valued in V and if $\beta_i = 0$ and S_3 is absent then theorem 1 is also true with (2.16) replaced by

$$D[w] \geqslant D[u]. \tag{6.2}$$

Proof. By successively using the definition of $D_e[w]$, setting $f_i = \Omega,_i$, recalling that $w_{i,i} = 0$, using Gauss' theorem, and using the fact that w satisfies (2.5) and (2.6) we obtain

$$\begin{aligned} D_e[w] - D[w] &= -2\int_V w_i f_i \, dV = -2\int_V w_i\, \Omega,_i \, dV = -2\int_V (w_i\,\Omega),_i \, dV \\ &= -2\int_{S_1+S_2} \Omega w_i n_i \, dS = -2\int_{S_1} \Omega g_i n_i \, dS - 2\int_{S_2} \Omega h \, dS. \end{aligned} \tag{6.3}$$

We note that the last expression in (6.3) is independent of w. We now use (6.3) in (2.16) to express $D_e[w]$ in terms of $D[w]$, and to express $D_e[u]$ in terms of $D[u]$. The surface integrals cancel and the result is (6.2), which proves the theorem.

According to theorem 7, when (6.1) holds and S_3 is absent the Stokes flow minimizes the dissipation rate in the class of continuous, piecewise continuously differentiable flows satisfying (2.3), (2.5)–(2.7) with $\beta_i = 0$. This shows that the flow is independent of the potential, since this characterization of the flow does not involve the potential. Of course the corresponding pressure is dependent upon the potential.

By using (6.3) in (3.10) we can reformulate theorem 2 as follows.

Theorem 8. If $f_i = \Omega,_i$ where Ω is continuous and single valued in V and if $\beta_i = 0$ and S_3 is absent, then theorem 2 is also true with (3.10) replaced by

$$H[\sigma_{ij}]+2\int_{S_1}\Omega g_i n_i\,dS+2\int_{S_2}\Omega h\,dS \leqslant H[\tau_{ij}[u]]+2\int_{S_1}\Omega g_i n_i\,dS+2\int_{S_2}\Omega h\,dS = D[u]. \quad (6.4)$$

It is helpful to observe that the sum of the two surface integrals in (6.4) will be unaffected by the addition of a constant to Ω provided

$$\int_{S_1} g_i n_i\,dS+\int_{S_2} h\,dS = 0. \quad (6.5)$$

This is a condition which must be satisfied if there is any flow u satisfying (2.3), (2.5) and (2.6) as we see by integrating (2.3) over V, applying Gauss' theorem and then using (2.5) and (2.6). It expresses the fact that the net mass flux out of V must vanish.

Theorem 7 can be reformulated in an interesting way by recalling that the rate of dissipation of energy into heat is proportional to the rate of entropy production. Therefore we can restate theorem 7 as follows.

Theorem 7′. A continuously differentiable Stokes flow in V, i.e. a solution of (2.3)–(2.7), with $\beta_i = 0$ and S_3 absent, has a smaller rate of entropy production than any continuous, piecewise continuously differentiable flow defined in V and satisfying (2.3), (2.5) and (2.6) provided the external force is derivable from a continuous single valued potential and that S_1 is not empty or that no rigid body motion of the fluid is possible in V with vanishing normal velocity on the surface of V.

This theorem is an instance of the 'principle of the minimum rate of entropy production' which is used in irreversible thermodynamics. Our theorem shows that this principle does not apply if the forces are not derivable from a continuous single valued potential. It also shows that the principle applies to the Stokes flow, which is a slow, slowly changing flow, and not to the solution of the unsteady non-linear Navier–Stokes equation. This is in accordance with the belief that the principle applies only to those steady motions which represent small departures from equilibrium.

For forces derivable from a potential we can also reformulate theorems 3 and 4 as follows.

Theorem 9. Suppose $f_i = \Omega,_i$ in V and $f_i^{(k)} = \Omega,_i$ in $V^{(k)}$ $(k=1,\ldots,N)$ where Ω is continuous and single valued in $V+\Sigma V^{(k)}$, $\beta_i = 0$, S_3 is absent and

$$F_i^{(k)} = \int_{s_k}\Omega n_i^{(k)}\,dS \quad (k=N+1,\ldots,N+M), \quad (6.6)$$

$$N_m^{(k)} = \int_{s_k}\Omega \epsilon_{imj} r_j^{(k)} n_i^{(k)}\,dS \quad (k=N+1,\ldots,N+M). \quad (6.7)$$

Then theorem 3 is true with (4.12) replaced by

$$D[\bar{u}]+\sum_{k=1}^{N} D^{(k)}[\bar{u}^{(k)}] \geqslant D[u]+\sum_{k=1}^{N} D^{(k)}[u^{k}], \tag{6.8}$$

and theorem 4 is true with (5.10) replaced by

$$\begin{aligned} H[\sigma_{ij}, \sigma_{ij}^{(k)}]+2\int_{S_1} \Omega n_i g_i \, dS+2\int_{S_2} \Omega h \, dS \\ \leqslant H[\tau_{ij}[u], \tau_{ij}[u^{(k)}]]+2\int_{S_1} \Omega n_i g_i \, dS+2\int_{S_2} \Omega h \, dS \\ = D[u]+\sum_{k=1}^{N} D^{(k)}[u^{(k)}]. \end{aligned} \tag{6.9}$$

This theorem enables us to obtain upper and lower bounds on the total dissipation rate of the fluid when the hypotheses are satisfied. The hypotheses (6.6) and (6.7) mean that the external force and torque on each solid particle are equal to the buoyant force and torque exerted on it by the fluid. Thus the solid particles must be neutrally buoyant.

An interesting application of theorem 1 is obtained by noting that a solution of the Navier–Stokes equation is a Stokes flow if it satisfies the condition

$$\partial u/\partial t+(u \, . \nabla) u = 0.$$

This is the case for steady laminar flow in a pipe or between parallel planes, which can occur if the only forces are potential forces. Thus in these cases the Navier–Stokes flow is also the Stokes flow $u(x)$. Consequently by theorem 1 these laminar flows have a smaller dissipation rate than any other flows, such as turbulent ones, having the same boundary values. Now the dissipation rate in a finite length of pipe is equal to the pressure drop between the ends of the pipe multiplied by the flux. The laminar flow and any other flows having the same boundary values have the same flux and therefore the laminar flow has the smallest pressure drop. Let us define the resistance coefficient for any flow as the ratio of the pressure drop per unit length to the flux. Then we obtain the following result, due to Thomas (1942).

Theorem 10. Parallel or laminar flow in a straight pipe has a smaller resistance coefficient than any other steady or unsteady incompressible flow having the same velocity distribution over the ends and walls of the pipe, provided any external forces acting derive from a continuous single valued potential.

Another useful consequence of theorem 1 is obtained by using for w the steady solution of the Navier–Stokes equations satisfying the same boundary conditions as the Stokes flow u, or any unsteady solution satisfying the same conditions. This yields

Theorem 11. Let u be the continuously differentiable Stokes flow in V satisfying (2.3)–(2.8). Let w be a continuous piecewise continuously differentiable steady or unsteady solution of the Navier–Stokes equation in V satisfying (2.3) and (2.5)–(2.8). Then

$$D_e[w] \geqslant D_e[u]. \tag{6.10}$$

The inequality holds in (6.10) if $w(x) \neq u(x)$ provided S_1 is not empty or provided no rigid body motion of the fluid is possible in V with vanishing normal velocity on S_2.

When $f_i = \Omega,_i$, theorem 7 yields

Theorem 12. If $f_i = \Omega,_i$ where Ω is continuous and single valued in V and if $\beta_i = 0$ and S_3 is absent then theorem 11 is true with (6.10) replaced by

$$D[w] \geqslant D[u]. \tag{6.11}$$

Next let us consider the steady motion of an object through a fluid. Any external force acting on the fluid is assumed to be derivable from a continuous single valued potential. In this case the dissipation rate is equal to the rate at which the object does work on the fluid. If the motion is purely translational this rate of work is the product of the speed and the drag force. If the motion is purely rotational, the rate of work is the product of the angular speed and the resistive torque, i.e. the component in the direction of the angular velocity, of the torque exerted by the object on the fluid. From theorem 12 we obtain

Theorem 13. The Stokes flows yield a smaller drag on a body in translation and a smaller resistive torque on a body in rotation in an incompressible fluid than do the corresponding Navier–Stokes flows, provided $\beta_i = 0$, S_3 is absent and any external forces on the fluid derive from a continuous single valued potential.

A special case of this result is given by Lamb (1945, p. 619) for the torque on a sphere in a concentric spherical container. The fact that the Oseen correction to the Stokes formula for the drag on a rigid sphere is positive is a consequence of theorem 13. So is the fact that the Brenner & Cox (1963) correction to the Stokes drag on an arbitrary object is positive. This theorem also applies to the drag on a gas bubble, provided the shape of the bubble is prescribed. Thus Moore's (1959) calculation of the Stokes drag on a spherical bubble provides a lower bound on the Navier–Stokes drag.

In the last three theorems results obtainable from the Stokes flow are compared with those given by the Navier–Stokes flow. A different type of application of the extremum principles is to obtain bounds on the dissipation rate or excess dissipation rate of the Stokes flow itself. This is important for those cases in which the Stokes flow has not been determined. It is most valuable when the dissipation rate has additional physical significance, such as when it is proportional to the resistive torque on a body in steady rotation, the flow resistance in a pipe or the drag on a body in steady translation. Some such applications have been made by Hill & Power (1956).

It has been pointed out to us by H. Brenner that the extremum principles can also be used to obtain bounds on all the elements of the Stokes translational, rotational and coupling resistance matrices, which relate the force and torque on an arbitrary rigid body to its linear and angular velocities (Happel & Brenner 1965, chapter 5). This can be done by obtaining bounds on the drag for six different directions of translational motion, and bounds on the torque for six different axes of rotational motion and then finding bounds on the dissipation rate for an appropriate number of cases of simultaneous translation and rotation. We shall not give any examples of such applications. Instead, in §§ 7 and 8 we shall give applications of our new extremum principles for flows containing particles to obtain bounds on the effective viscosity and sedimentation velocity of suspensions.

All the previous results can be applied to spatially periodic flows by choosing as the domain V one period cell of the flow. The comparison flows must also have the same periodicity. Then all relevant surface integrals over the surface of a cell vanish by periodicity.

7. Viscosity of a suspension

By a suspension we mean a viscous fluid containing a collection of solid particles, droplets of a different liquid, or gas bubbles. A large-scale motion of a suspension is a motion with a scale large compared to the particle size and to the interparticle spacing. For such motions of an isotropic suspension it often suffices to treat the suspension as a uniform fluid having some density ρ_s and some viscosity coefficient μ_s†. For a nonisotropic suspension μ_s depends upon the direction of shear. We call ρ_s and μ_s the density and viscosity of the suspension. Our aim is to determine μ_s. In doing so we assume that there are no external forces or torques acting so that $f = f^{(k)} = 0$ and $F^{(k)} = N^{(k)} = 0$.

We define μ_s as the viscosity of that uniform incompressible fluid which dissipates energy at the same rate as the suspension when both the uniform fluid and the suspension occupy identical domains V and satisfy the same boundary conditions. To eliminate the dependence of μ_s on the size and shape of V we let V become infinite and require only that the average energy dissipation rate per unit volume be the same for the two flows. For simplicity we choose the boundary condition on the surface S_1 of V to be

$$u_i = \alpha_{ij} x_j, \quad x \text{ on } S_1; \qquad \alpha_{ii} = 0, \quad \alpha_{ij} = \alpha_{ji}. \tag{7.1}$$

Here α_{ij} is a constant. When V is filled with a uniform fluid the resulting flow is a uniform shear flow with constant strain rate $e_{ij} = \alpha_{ij}$. The dissipation rate per unit volume is $2\mu_s \alpha_{ij} \alpha_{ij}$. When V is filled with the suspension the dissipation rate is denoted by D. Then with v denoting the volume of V, our definition yields

$$\mu_s = \lim_{v\to\infty} \frac{D}{2\alpha_{ij}\alpha_{ij} v}. \tag{7.2}$$

When the motion of the suspension is a Stokes flow, i.e. a solution of (4.0)–(4.10), our extremum principles enable us to obtain upper and lower bounds on D. Upon using them in (7.2) we get upper and lower bounds on μ_s,

$$\lim_{v\to\infty} \frac{H[\sigma_{ij}, \sigma_{ij}^{(k)}]}{2\alpha_{ij}\alpha_{ij} v} \leqslant \mu_s \leqslant \lim_{v\to\infty} \frac{D[\bar{u}, \bar{u}^{(k)}, \bar{U}^{(k)} \bar{\omega}^{(k)}]}{2\alpha_{ij}\alpha_{ij} v}. \tag{7.3}$$

In order to use (7.3) to obtain explicit bounds on μ_s, we must find admissible tensors σ_{ij} and $\sigma_{ij}^{(k)}$ and motions $\bar{u}, \bar{u}^{(k)}$, $\bar{U}^{(k)}$ and $\bar{\omega}^{(k)}$.

We shall now describe one way of constructing admissible functions. First we determine the distance from the reference point in the kth particle to the nearest reference point of any other particle, and denote it by $2b_k$. Then for each k we draw a sphere of radius b_k about the reference point of particle k. By construction

† H. Brenner has pointed out that this is so only at sufficiently low concentrations. For higher concentrations the state of stress cannot generally be described by using a single coefficient.

these spheres do not overlap. We assume that each particle is entirely inside the sphere about its reference point. In the region outside all the spheres we define $\bar{u}$ and σ_{ij} by

$$\bar{u}_i = \alpha_{ij}\, x_j, \tag{7.4}$$

$$\sigma_{ij} = 2\mu\alpha_{ij}. \tag{7.5}$$

From (7.4) we find that the dissipation rate per unit volume in this region is $2\mu\alpha_{ij}\,\alpha_{ij}$. The volume of this region is $v - \sum_k 4\pi b_k^3/3$ where the sum is taken over all the particles in V, provided we ignore the parts of spheres which lie outside V. The sum of these parts is proportional to the area of the surface of V and therefore it will vanish compared to v in the limit of v becoming infinite, which justifies neglecting it. We have not yet defined the flow inside the kth sphere, but we shall denote by $D^{(k)}$ its dissipation rate. Then we have

$$D[\bar{u}, \bar{u}^{(k)}, \bar{U}^{(k)}, \bar{\omega}^{(k)}] = 2\mu\alpha_{ij}\,\alpha_{ij}(v - \sum_k 4\pi b_k^3/3) + \sum_k D^{(k)}. \tag{7.6}$$

Before using (7.6) in (7.3), let us evaluate $H[\sigma_{ij}, \sigma_{ij}^{(k)}]$ by using (7.5) in the region outside the spheres. From (5.1) we find

$$H[\sigma_{ij}, \sigma_{ij}^{(k)}] = -2\mu\alpha_{ij}\,\alpha_{ij}(v - \sum_k 4\pi b_k^3/3) + 4\mu\alpha_{ij}\int_{S_1} g_i\, n_j\, dS + \sum_k H^{(k)}. \tag{7.7}$$

Here we have introduced $H^{(k)}$ which is the contribution to H from B_k, the interior of the sphere containing the kth particle. It has different forms for fluid and solid particles, which are the following:

$$H^{(k)} = -\frac{1}{2\mu}\int_{B_k - V_k}(\sigma_{ij} - \tfrac{1}{3}\sigma_{nn}\,\delta_{ij})^2\, dV - \frac{1}{2\mu^{(k)}}\int_{V_k}(\sigma_{ij}^{(k)} - \tfrac{1}{3}\sigma_{nn}^{(k)}\,\delta_{ij})^2\, dV \quad (k = 1, \ldots, N), \tag{7.8}$$

$$H^{(k)} = -\frac{1}{2\mu}\int_{B_k - V_k}(\sigma_{ij} - \tfrac{1}{3}\sigma_{nn}\,\delta_{ij})^2\, dV \quad (k = N+1, \ldots, M). \tag{7.9}$$

In (7.7) g_i is the ith component of the prescribed velocity on the outer surface of V, which is $\alpha_{il}\, x_l$. By using this value in the surface integral in (7.7) and applying Gauss' theorem we can evaluate the integral as follows:

$$4\mu\alpha_{ij}\int_{S_1} g_i\, n_j\, dS = 4\mu\alpha_{ij}\,\alpha_{il}\int_{S_1} x_l\, n_j\, dS = 4\mu\alpha_{ij}\,\alpha_{il}\int_V x_{l,j}\, dV$$
$$= 4\mu\alpha_{ij}\,\alpha_{il}\,\delta_{lj}\, v = 4\mu\alpha_{ij}\,\alpha_{ij}\, v. \tag{7.10}$$

When (7.10) is used in (7.7), (7.7) becomes

$$H[\sigma_{ij}, \sigma_{ij}^{(k)}] = 2\mu\alpha_{ij}\,\alpha_{ij}(v + \sum_k 4\pi b_k^3/3) + \sum_k H^{(k)}. \tag{7.11}$$

We now substitute (7.6) and (7.11) in (7.3) to obtain

$$1 + \lim_{v\to\infty}\frac{1}{v}\sum_k\left(\frac{H^{(k)}}{2\mu\alpha_{ij}\,\alpha_{ij}} + \frac{4\pi}{3}b_k^3\right) \leqslant \frac{\mu_s}{\mu} \leqslant 1 + \lim_{v\to\infty}\frac{1}{v}\sum_k\left(\frac{D^{(k)}}{2\mu\alpha_{ij}\,\alpha_{ij}} - \frac{4\pi}{3}b_k^3\right). \tag{7.12}$$

In defining σ_{ij} inside the sphere of radius b_k around particle k, we must make $n_i\,\sigma_{ij}$ continuous across the spherical surface, where n_i is the normal to this surface so $n_i\,\sigma_{ij} = 2\mu n_i\,\alpha_{ij}$ on $r^{(k)} = b_k$. Similarly, in defining $\bar{u}$ inside the sphere, we must make $\bar{u}$ continuous and therefore $\bar{u} = \alpha_{ij} x_j$ on the surface $r^{(k)} = b_k$. The best bounds obtainable from (7.12) are those for which $H^{(k)}$ is as large as possible and $D^{(k)}$ as small as possible, subject to the boundary conditions just mentioned. According to theorem 7, the smallest value of $D^{(k)}$ is given by the solution of the slow motion problem (4.0)–(4.10) inside the sphere of radius b_k with $\bar{u} = \alpha_{ij} x_j$ on its surface. Similarly, by theorem 8 the largest value of $H^{(k)}$ is given by the solution of (4.0)–(4.10) inside B_k with $n_j \tau_{ij}[u] = 2\mu n_j \alpha_{ij}$ on its surface. These problems are simpler than the original problem because each sphere contains only one particle. Of course poorer bounds can be obtained more easily by using any admissible functions rather than solving these problems.

One case in which these problems can be solved is that of spherical particles, which we shall now consider. Let $D(a,b) = \min D^{(k)}$ and $H(a,b) = \max H^{(k)}$ when the sphere of radius b surrounds a concentric spherical particle of radius a. Then $D(a,b)$ and $H(a,b)$ are determined by the solutions of the two slow-motion problems we have just described. They are given by (A 12) of appendix A and (B 4) of appendix B, respectively. When we use $D(a,b)$ and $H(a,b)$ in (7.12), we can express the limits in terms of the number density $g_1(a,b)$ of particles of radius a with nearest neighbour at distance $2b$. Since $g_1(a,b) = 0$ for $b < a$, (7.12) becomes

$$\int_0^\infty \int_a^\infty \left[\frac{H(a,b)}{2\mu\alpha_{ij}\alpha_{ij}} + \frac{4\pi}{3} b^3\right] g_1(a,b)\, db\, da \leqslant \frac{\mu_s}{\mu} - 1$$
$$\leqslant \int_0^\infty \int_a^\infty \left[\frac{D(a,b)}{2\mu\alpha_{ij}\alpha_{ij}} - \frac{4\pi}{3} b^3\right] g_1(a,b)\, db\, da. \quad (7.13)$$

For low concentration of particles, $g_1(a,b)$ is practically zero unless $a \ll b$. Then $H(a,b)$ and $D(a,b)$ in the integrands of (7.13), can be expanded in powers of $\lambda = a/b$ to yield

$$\frac{(5\eta+2)}{2(1+\eta)} \int_0^\infty \frac{4\pi a^3}{3} \int_a^\infty g_1(a,b)\, db\, da + \ldots \leqslant \frac{\mu_s}{\mu} - 1$$
$$\leqslant \frac{(5\eta+2)}{2(1+\eta)} \int_0^\infty \frac{4\pi a^3}{3} \int_a^\infty g_1(a,b)\, db\, da + \ldots. \quad (7.14)$$

From the definition of $g_1(a,b)$, the integrals appearing in (7.14) represent the total volume of particles per unit volume of fluid, which is just the volume concentration c. Noting that the first term is the same on both sides of the inequality in (7.14), we have†

$$\frac{\mu_s}{\mu} = 1 + \frac{(2+5\eta)}{2(1+\eta)} c + \ldots. \quad (7.15)$$

The result (7.15) agrees with that obtained by Taylor (1932) for a low concentration suspension of fluid spheres. If we let $\eta \to \infty$, it becomes Einstein's (1906) result for a low concentration suspension of solid spheres. Our derivation, which applies to any distribution of spheres, is actually a proof of (7.15), whereas the previous derivations were approximate calculations and not proofs.

† For particles of any shape we can show that $\mu_s/\mu = 1 + Kc + \ldots$ where K depends upon η and the shape.

If we now specialize to the case of identical spherical particles of radius a, then we set $g_1(a',b) = g(b)\,\delta(a'-a)$ and (7.13) becomes

$$\int_0^\infty \left[\frac{H(a,b)}{2\mu\alpha_{ij}\alpha_{ij}} + \frac{4\pi}{3}b^3\right] g(b)\,db \leqslant \frac{\mu_s}{\mu} - 1 \leqslant \int_0^\infty \left[\frac{D(a,b)}{2\mu\alpha_{ij}\alpha_{ij}} - \frac{4\pi}{3}b^3\right] g(b)\,db. \quad (7.16)$$

The function $g(b)$ is determined by the spatial distribution of particles. For independent point particles of number density n, g is given by the Poisson distribution

$$g(b) = 16\pi n b^2 e^{-32nb^3/3}. \quad (7.17)$$

If the particles are small compared to their average separation, this should be a good approximation to $g(b)$ for $b \gg a$. On the other hand, if the particles are situated on a lattice of minimum spacing $2b$ and number density n, then

$$g(b') = n\delta(b'-b). \quad (7.18)$$

The two distributions (7.17) and (7.18) represent opposite extremes of the possible distributions. For a simple cubic lattice $n = 1/8b^3$ and (7.16) becomes, when (7.18) is used in it,

$$\frac{H(a,b)}{16b^3\mu\alpha_{ij}\alpha_{ij}} + \frac{\pi}{6} \leqslant \frac{\mu_s}{\mu} - 1 \leqslant \frac{D(a,b)}{16b^3\mu\alpha_{ij}\alpha_{ij}} - \frac{\pi}{6}. \quad (7.19)$$

By substituting into (7.19) the values of $D(a,b)$ and $H(a,b)$ given in appendices A and B respectively, we get the following upper and lower bounds on μ_s/μ for a simple cubic lattice of spheres:

$$\frac{\mu_s}{\mu} \leqslant 1 - \frac{\pi\lambda^3[5(1-\eta)\lambda^7 + (5\eta+2)]}{3[4(1-\eta)\lambda^{10} + 5(5\eta-2)\lambda^7 - 42\eta\lambda^5 + 5(5\eta+2)\lambda^3 - 4(1+\eta)]}, \quad (7.20)$$

$$\frac{\mu_s}{\mu} \geqslant 1 + \frac{\pi\lambda^3[-80(1-\eta)\lambda^7 + 19(2+5\eta)]}{6[-48(1-\eta)\lambda^{10} - 40(2-5\eta)\lambda^7 - 336\eta\lambda^5 + 45(2+5\eta)\lambda^3 + 38(1+\eta)]}. \quad (7.21)$$

Here $\lambda = a/b$ and $\eta = \mu_1/\mu$, where μ_1 is the viscosity coefficient of the fluid inside the spheres. Graphs of these bounds are shown in figure 1 as functions of $\pi\lambda^3/6$ for several values of η.

The volume concentration of particles for the cubic lattice is given by

$$c = 4\pi a^3/3(8b^3) = \pi\lambda^3/6.$$

Thus c varies from zero at $\lambda = 0$ to $\pi/6$ at $\lambda = 1$, when neighbouring spheres just touch. For low concentration, $c \ll 1$, (7.20) and (7.21) can be expanded to yield

$$1 + \frac{(5\eta+2)}{2(1+\eta)}c - \frac{135(5\eta+2)^2}{38\pi(1+\eta)^2}c^2 + \ldots \leqslant \frac{\mu_s}{\mu} \leqslant 1 + \frac{(5\eta+2)}{2(1+\eta)}c + \frac{15(5\eta+2)^2}{4\pi(1+\eta)^2}c^2 + \ldots. \quad (7.22)$$

For high concentration, $c - \pi/6 \ll 1$, (7.18) and (7.19) become

$$1 + \frac{\pi}{6} - \frac{7\pi}{175\eta} + \ldots \leqslant \frac{\mu_s}{\mu} \leqslant \frac{\pi^4}{480}\left(\frac{\pi}{6} - c\right)^{-3} + \ldots. \quad (7.23)$$

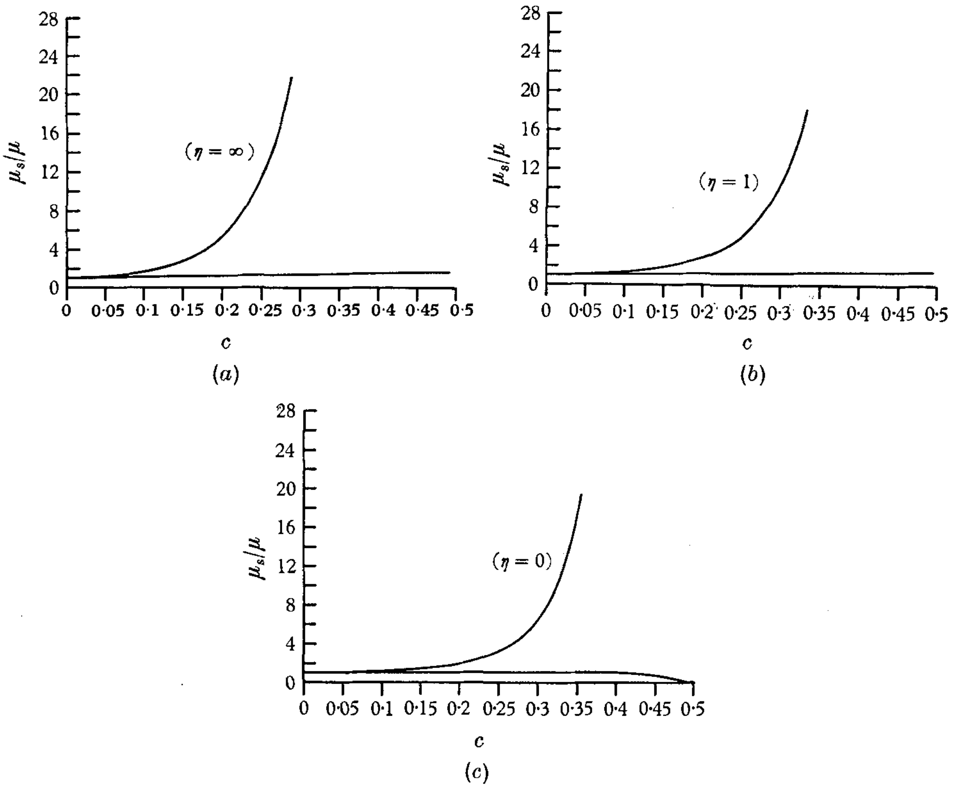

FIGURE 1. Upper and lower bounds on μ_s/μ as a function of concentration c for a suspension of spherical particles in an incompressible fluid. All particles have radius a and their centres are located on a simple cubic lattice of spacing $2b$. The bounds are given by equations (7.20) and (7.21) in which $\lambda = a/b$ and $\eta = \mu_1/\mu$, where μ_1 is the viscosity of the fluid of the particles and μ is the viscosity of the surrounding fluid. The case $\eta = \infty$ corresponds to rigid particles and $\eta = 0$ to gas bubbles. The concentration $c = \pi\lambda^3/6$ varies from zero to $\pi/6$ when the spheres touch each other.

8. Sedimentation velocity

Consider a collection of particles settling or rising in a viscous fluid under the influence of gravity. It is convenient to introduce an average particle velocity U which we call the settling or sedimentation velocity. To define U we consider W_t, the time rate of change of potential energy W of the fluid and particles in some domain V. It is given by

$$W_t = \sum_k (m_k - \rho V_k)\, g U_3^{(k)}. \tag{8.1}$$

Here m_k, V_k and $U_3^{(k)}$ denote the mass, volume and vertical velocity of the centroid of particle k, ρ is the fluid density and g is the acceleration of gravity. The sum is over all particles in the domain V. Now we define U as that single velocity which would yield the same rate of change of W. However, to eliminate the dependence of U upon the size and shape of V, we consider the limit as V becomes infinite. Thus we define U by

$$U = \lim_{V\to\infty} \frac{\sum_k (m_k - \rho V_k)\, U_3^{(k)}}{\sum_k (m_k - \rho V_k)} = \lim_{V\to\infty} \frac{W_t}{g \sum_k (m_k - \rho V_k)}. \tag{8.2}$$

In order to determine U we observe that in a Stokes flow of a suspension of fluid drops, the rate of change of potential energy W_t is equal to minus the rate of energy dissipation

$$W_t = -D[u] - \sum_{k=1}^{N} D^{(k)}[u^{(k)}]. \tag{8.3}$$

Furthermore, suppose u is given on the boundary S_1 of V, so that S_2 and S_3 are absent, and that all the particles are fluid drops. Then from (4.11) we have

$$D_e[u, u^{(k)}, U^{(k)}, \omega^{(k)}] = D[u] + \sum_{k=1}^{N} D^{(k)}[u^{(k)}] + 2W_t. \tag{8.4}$$

Now (8.3) and (8.4) yield

$$D_e[u, u^{(k)}, U^{(k)}, \omega^{(k)}] = W_t. \tag{8.5}$$

By combining (8.2) and (8.5) we obtain

$$U = \lim_{V\to\infty} \frac{D_e[u, u^{(k)}, U^{(k)}, \omega^{(k)}]}{g\sum_k (m_k - \rho V_k)}. \tag{8.6}$$

It is convenient to rewrite (8.6) in terms of the volume v of V in the form

$$gU \lim_{v\to\infty} \frac{1}{v}\sum_k (m_k - \rho V_k) = \lim_{v\to\infty} \frac{1}{v} D_e(u, u^{(k)}, U^{(k)}, \omega^{(k)}]. \tag{8.7}$$

Then from the minimum principle, theorem 3, we have

$$gU \lim_{v\to\infty} \frac{1}{v}\sum_k (m_k - \rho V_k) \leqslant \lim_{v\to\infty} \frac{1}{v} D_e[\overline{u}, \overline{u}^{(k)}, \overline{U}^{(k)}, \overline{\omega}^{(k)}]. \tag{8.8}$$

Thus (8.8) yields an upper or lower bound on U according as the coefficient of U is positive or negative.

To construct an admissible motion, $\overline{u}, \overline{u}^{(k)}, \overline{U}^{(k)}, \overline{\omega}^{(k)}$ to use in (8.8), we proceed as in § 7. Thus for each k we introduce a sphere of radius b_k around the centroid of particle k, where b_k is half the distance to the nearest particle centroid. Then we set $\overline{u} = 0$ outside all these spheres and denote by $D_e^{(k)}$ the excess dissipation rate of the still undefined flow within the sphere of radius b_k. In terms of $D_e^{(k)}$, (8.8) becomes

$$gU \lim_{v\to\infty} \frac{1}{v}\sum_k (m_k - \rho V_k) \leqslant \lim_{v\to\infty} \frac{1}{v}\sum_k D_e^{(k)}. \tag{8.9}$$

The smallest value of the right side of (8.9) is obtained by choosing the Stokes flow as the flow within the sphere of radius b_k. Since $\overline{u}$ must be continuous, this flow must satisfy the condition $\overline{u} = 0$ at $r = b_k$. Its determination is simpler than that of the original flow because this sphere contains only one particle.

When the particle is a sphere of radius a, the Stokes flow within the concentric sphere of radius b can be determined explicitly. This is done in appendix C and the excess dissipation rate for it, denoted by $D_e(a, b)$, is found. It is given by (C 13). We now assume that the particles are spheres and take the limit in (8.9) to obtain

$$\frac{4\pi g}{3} U \int_0^\infty a^3[\rho_1(a) - \rho]\,\hat{g}(a)\,da \leqslant \int_0^\infty \int_a^\infty D_e(a, b)\, g_1(a, b)\, db\, da. \tag{8.10}$$

Here $g_1(a,b)$ is the number density of particles of radius a with nearest neighbour at distance $2b$, $\hat{g}(a)$ is the number density of particles of radius a and $\rho_1(a)$ is the density of a particle of radius a.

For a simple cubic lattice of identical particles of radius a and spacing $2b$, (8.10) becomes

$$\tfrac{4}{3}\pi g a^3(\rho_1-\rho)\,U \leqslant D_e(a,b). \tag{8.11}$$

By using (C 13) of appendix C in (8.11) we obtain

$$\frac{U}{U_0} \geqslant \left[\frac{2+3\eta}{1+\eta}\right]\frac{(1-\lambda)^3[4(1-\eta)\lambda^3+3(2-\eta)\lambda^2+3(2+\eta)\lambda+4(1+\eta)]}{4[3(1-\eta)\lambda^5+(2+3\eta)]}. \tag{8.12}$$

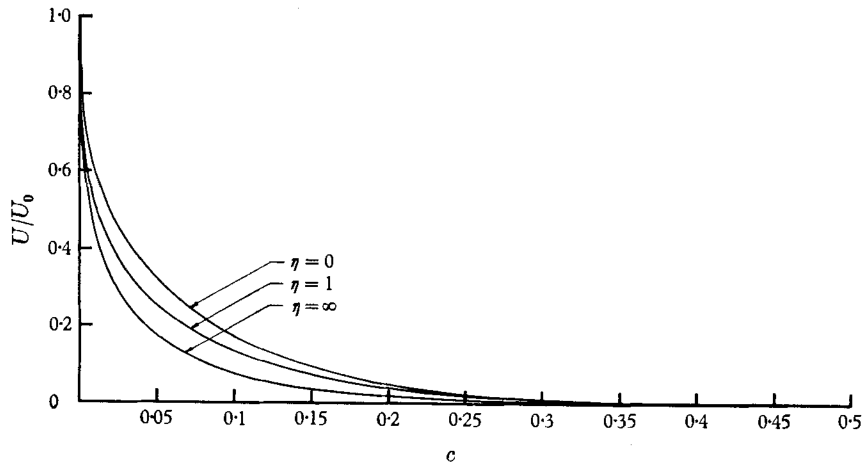

FIGURE 2. Lower bounds on U/U_0 as a function of concentration c for a suspension of spherical particles in an incompressible fluid. All particles have radius a and their centres are located on a simple cubic lattice of spacing $2b$. The bounds are given by equation (8.12) in which $\lambda = a/b$ and $\eta = \mu_1/\mu$, where μ_1 is the viscosity of the fluid in the particles and μ is the viscosity of the surrounding fluid. The case $\eta = \infty$ corresponds to rigid particles and $\eta = 0$ to gas bubbles. The concentration $c = \pi\lambda^3/6$ varies from zero to $\pi/6$ when the spheres touch each other.

Here U_0 is the terminal velocity of a single liquid sphere of radius a and density ρ_1 falling in a liquid of density ρ. It is given by

$$U_0 = -\frac{2ga^2(\rho_1-\rho)(1+\eta)}{3\mu(2+3\eta)}. \tag{8.13}$$

The minus sign in the definition of U_0 occurs because upward velocity is positive. This sign is responsible for the reversal of the inequality in going from (8.11) to (8.12). From (8.12) we find that U has the same sign as U_0. Graphs of the lower bound (8.12) are shown in figure 2 as a function of $\pi\lambda^3/6$ for several values of η.

When the spheres just touch, $\lambda = 1$ and the lower bound in (8.12) is zero. When λ is small, and the concentration $c = \pi\lambda^3/6$ is introduced, the lower bound becomes

$$\frac{U}{U_0} \geqslant 1-\frac{3(2+3\eta)}{4(1+\eta)}\left(\frac{6c}{\pi}\right)^{\frac{1}{3}}+\ldots. \tag{8.14}$$

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Appendix A. Slow shear flow of two concentric liquid spheres with surface velocity prescribed

Consider an incompressible liquid sphere of radius b and viscosity coefficient μ containing a concentric incompressible liquid sphere of radius $a < b$ and viscosity coefficient μ_1. We wish to find the slow steady shear flow of these liquids due to a prescribed velocity distribution at the outer boundary $r = b$. The cases in which the inner sphere is a solid or a vacuum are included as the limiting cases $\mu_1/\mu = \infty$ and $\mu_1/\mu = 0$ respectively. At the outer boundary we require that

$$u_i = \alpha_{ij} x_j, \quad r = b. \tag{A 1}$$

Here α_{ij} is a constant tensor satisfying

$$\alpha_{ij} = \alpha_{ji}; \quad \alpha_{ii} = 0. \tag{A 2}$$

The equations for the determination of u and p are:

$$u_{i,i} = 0, \quad r \leqslant b, \tag{A 3}$$

$$\mu \Delta u_i - p_{,i} = 0, \quad a \leqslant r \leqslant b, \tag{A 4}$$

$$\mu_1 \Delta u_i - p_{,i} = 0, \quad 0 \leqslant r \leqslant a. \tag{A 5}$$

At $r = a$, u_i must be continuous, its radial component must vanish, the tangential components of normal stress, $n_i \tau_{ij}$, must be continuous and u must be continuous for $0 \leqslant r \leqslant b$. In addition, the over-all translational and rotational velocities of the inner spherical surface must be chosen so that there is no net force or torque on that surface.

We have solved (A 3)–(A 5) for u and p subject to (A 1), (A 2) and the above continuity conditions by using the method described by Lamb (1945). In doing so we assumed that the inner spherical surface did not move, in view of the symmetry of (A 1) and (A 2). Our solution confirms this, since it yields no net force or torque on the fluid in $r \leqslant a$. The results are

$$u_i(\mathbf{x}) = \frac{\phi(\lambda)}{\Delta(\lambda;\eta)}\left[\left(\frac{-5r^3}{a^2}+3r\right)\alpha_{ij}n_j + \frac{2r^3}{a^2}\alpha_{jl}n_j n_l n_i\right] \quad (0 \leqslant r \leqslant a), \tag{A 6}$$

$$u_i(\mathbf{x}) = \frac{1}{\Delta(\lambda;\eta)}\left[\phi_1(\lambda;\eta)\,r - \frac{5r^3}{a^2}\phi_2(\lambda;\eta) + \frac{4a^5}{r^4}\phi_3(\lambda;\eta)\right]\alpha_{ij}n_j$$
$$+\frac{1}{\Delta(\lambda;\eta)}\left[\frac{2r^3}{a^2}\phi_2(\lambda;\eta) + \frac{2a^3}{r^2}\phi_4(\lambda;\eta) - \frac{10a^5}{r^4}\phi_3(\lambda;\eta)\right]\alpha_{jl}n_j n_l n_i \quad (a \leqslant r \leqslant b), \tag{A 7}$$

$$p(\mathbf{x}) = p_0 - \frac{21 r^2 \mu_1 \phi(\lambda)\,\alpha_{ij}n_i n_j}{a^2\,\Delta(\lambda;\eta)} \quad (0 \leqslant r \leqslant a), \tag{A 8}$$

$$p(\mathbf{x}) = p_0 - \frac{\mu}{\Delta(\lambda;\eta)}\left[\frac{21r^2}{a^2}\phi_2(\lambda;\eta) - \frac{4a^3}{r^3}\phi_4(\lambda;\eta)\right]\alpha_{ij}n_i n_j \quad (a \leqslant r \leqslant b). \tag{A 9}$$

Here, $\lambda = a/b$, $\eta = \mu_1/\mu$, p_0 is an arbitrary constant, $r = |x|$, $n_i = x_i/r$ is the ith component of a unit vector and the other quantities are defined by

$$\left.\begin{aligned}
\phi(\lambda) &= 5\lambda^7 - 7\lambda^5 + 2,\\
\phi_1(\lambda;\eta) &= -[5(2-5\eta)\lambda^7 + 21\eta^5 + 4(1+\eta)],\\
\phi_2(\lambda;\eta) &= \lambda^5[5\eta\lambda^2 - (2+5\eta)],\\
\phi_3(\lambda;\eta) &= (1-\eta)\lambda^5 + \eta,\\
\phi_4(\lambda;\eta) &= 5(1-\eta)\lambda^7 + 2 + 5\eta,\\
\Delta(\lambda;\eta) &= 4(1-\eta)\lambda^{10} - 5(2-5\eta)\lambda^7 - 42\eta\lambda^5 + 5(2+5\eta)\lambda^3 - 4(1+\eta).
\end{aligned}\right\} \tag{A 10}$$

To evaluate the dissipation rate $D(a,b)$ for this flow, we first convert the volume integrals defining D to the surface integral

$$D(a,b) = \int_{r=b} u_i \tau_{ij}[u]\, n_j\, dS. \tag{A 11}$$

This equation states that the dissipation rate is equal to the power supplied at the outer surface $r = b$. We now use (A 7) and (A 9) to evaluate the integrand in (11) at $r = b$ and then integrate to obtain

$$D(a,b) = \frac{8\pi\mu b^3 \alpha_{ij}\alpha_{ij}[6(\eta-1)\lambda^{10} + 5(5\eta-2)\lambda^7 - 42\eta\lambda^5 + 3(5\eta+2)\lambda^3 - 4(1+\eta)]}{3[4(1-\eta)\lambda^{10} + 5(5\eta-2)\lambda^7 - 42\eta\lambda^5 + 5(5\eta+2)\lambda^3 - 4(1+\eta)]}. \tag{A 12}$$

Appendix B. Slow shear flow of two concentric liquid spheres with normal stress prescribed

We now reconsider the problem of appendix A with the boundary condition (A 1) replaced by

$$n_j \tau_{ij}[u] = 2\mu n_j \alpha_{ij}, \quad r = b. \tag{B 1}$$

All the other conditions of the problem are the same. Thus the normal stress is prescribed on the outer surface instead of the velocity.

We have solved this problem by the same method as before. Again we assumed that the interface $r = a$ does not move and verified that the resulting solution yields no net force or torque on the fluid inside this surface. The results are given by (A 6)–(A 9) where the definitions in (A 10) are replaced by

$$\left.\begin{aligned}
\phi(\lambda) &= 40\lambda^7 - 56\lambda^5 - 19,\\
\phi_1(\lambda;\eta) &= -2[20(2-5\eta)\lambda^7 + 84\eta\lambda^5 - 19(1+\eta)],\\
\phi_2(\lambda;\eta) &= 8\lambda^5[5\eta\lambda^2 - (2+5\eta)],\\
\phi_3(\lambda;\eta) &= \tfrac{1}{2}[16\lambda^5(1-\eta) - 19\eta],\\
\phi_4(\lambda;\eta) &= \tfrac{1}{2}[80\lambda^7(1-\eta) - 19(2+5\eta)],\\
\Delta(\lambda;\eta) &= -48(1-\eta)\lambda^{10} - 40(2-5\eta)\lambda^7 - 336\eta\lambda^5 + 45(2+5\eta)\lambda^3 + 38(1+\eta).
\end{aligned}\right\} \tag{B 2}$$

To compute $H(a,b)$ we make use of (5.2). In view of the fact that τ_{ij} is the stress tensor of a Stokes flow, (5.2) applies and shows that $H(a,b)$ equals minus the dissipation rate because $F_i = 0$ and S_2 is absent. Then we use the equality of the dissipation rate and the power supplied at the surface to write

$$H(a,b) = -\int_{r=b} u_i \tau_{ij}[u] n_j dS. \tag{B 3}$$

Upon using (A 7) and (A 9) together with (B 2) we obtain

$$H(a,b) = \frac{-8\pi\mu\alpha_{ij}\alpha_{ij}b^3[160(1-\eta)\lambda^{10} - 200(2-5\eta)\lambda^7 - 1680\eta\lambda^5 + 130(2+5\eta)\lambda^3 + 190(1+\eta)]}{15[-48(1-\eta)\lambda^{10} - 40(2-5\eta)\lambda^7 - 336\eta\lambda^5 + 45(2+5\eta)\lambda^3 + 38(1+\eta)]}. \tag{B 4}$$

Appendix C. Slow uniform flow of two concentric liquid spheres under gravity with zero surface velocity

We now wish to consider the slow steady flow of the two liquids described in appendix A satisfying the following equations:

$$u_{i,i} = 0, \qquad r \leqslant b, \tag{C 1}$$

$$\mu\Delta u_i - p_{,i} = \rho g\,\delta_{i3}, \quad a \leqslant r \leqslant b, \tag{C 2}$$

$$\mu_1\Delta u_i - p_{,i} = \rho_1 g\,\delta_{i3}, \quad 0 \leqslant r \leqslant a. \tag{C 3}$$

In (C 2) and (C 3) the inhomogeneous terms represent gravitational forces acting in the negative z direction, where ρ and ρ_1 are the densities of the two liquids and g is the acceleration of gravity.

At the outer boundary we require that

$$u_i = 0, \quad r = b. \tag{C 4}$$

At $r = a$, u_i and the tangential component of normal stress $n_i\tau_{ij}$ must be continuous. In addition the radial component of u_i must satisfy

$$n_i u_i = U n_3, \quad r = a, \tag{C 5}$$

where the over-all translational speed U is to be determined from the force balance equation

$$\int_{r\leqslant a} -\rho_1 g\delta_{i3}\,dV + \int_{r=a} n_j\tau_{ij}[u^+]\,dS = 0. \tag{C 6}$$

In (C 6), u^+ is the velocity of the fluid in $a \leqslant r \leqslant b$.

We have solved this problem by the same method as before, under the assumption that there is no over-all rotational velocity of the inner sphere. We have verified that the resulting solution yields no net torque on the fluid inside this surface. The results are

$$u_i = \frac{-U\phi(\lambda)}{\Delta(\lambda;\eta)}\left[\left(\frac{2r^2}{a^2}-1\right)\delta_{i3} - \frac{r^2}{a^2}n_i n_3\right] + U\delta_{i3} \quad (0 \leqslant r \leqslant a), \tag{C 7}$$

$$\begin{aligned} u_i = {} & \frac{-U\delta_{i3}}{\Delta(\lambda;\eta)}\left[\phi_1(\lambda;\eta) + \frac{r^2}{a^2}\phi_2(\lambda;\eta) + \frac{a}{r}\phi_3(\lambda;\eta) + \frac{a^3}{r^3}\phi_4(\lambda;\eta)\right] \\ & - \frac{U}{\Delta(\lambda;\eta)}\left[\frac{-r^2}{2a^2}\phi_2(\lambda;\eta) + \frac{a}{r}\phi_3(\lambda;\eta) - \frac{3a^3}{r^3}\phi_4(\lambda;\eta)\right] n_i n_3 + U\delta_{i3} \quad (a \leqslant r \leqslant b), \end{aligned} \tag{C 8}$$

$$p = p_0 - \frac{10\mu_1 U\phi(\lambda)x_3}{a^2\Delta(\lambda;\eta)} - \rho_1 g x_3 \quad (0 \leqslant r \leqslant a), \tag{C 9}$$

$$p = p_0 - \frac{\mu U n_3}{a\Delta(\lambda;\eta)}\left[\frac{5r}{a}\phi_2(\lambda;\eta) + \frac{2a^2}{r^2}\phi_3(\lambda;\eta)\right] - \rho g x_3 \quad (a \leqslant r \leqslant b), \tag{C 10}$$

$$U = \frac{a^2 g(\rho_1 - \rho)\Delta(\lambda;\eta)}{6\mu\phi_3(\lambda;\eta)}. \tag{C 11}$$

Here λ and η are defined as before and the other quantities are given by

$$\left.\begin{aligned} \phi(\lambda) &= 3(3\lambda^5 - 5\lambda^2 + 2), \\ \phi_1(\lambda;\eta) &= 3[3(2-3\eta)\lambda^5 + 5\eta\lambda^3 + 4(1+\eta)], \\ \phi_2(\lambda;\eta) &= 6\lambda^3[3\eta\lambda^2 - (2+3\eta)], \\ \phi_3(\lambda;\eta) &= 3[3(\eta-1)\lambda^5 - (2+3\eta)], \\ \phi_4(\lambda;\eta) &= 3[(\eta-1)\lambda^3 - \eta], \\ \Delta(\lambda;\eta) &= 3(1-\lambda)^3[4(1-\eta)\lambda^3 + 3(2-\eta)\lambda^2 + 3(2+\eta)\lambda + 4(1+\eta)]. \end{aligned}\right\} \tag{C 12}$$

These results agree with those given by Happel & Brenner (1965, pp. 130–3).

From (C 7)–(C 12) we can compute $D(a,b)$, the generalized dissipation rate, defined by (4.11), and obtain

$$D(a,b) = \frac{6\pi g^2 a^5(\rho_1-\rho)^2(1-\lambda)^3[4(1-\eta)\lambda^3 + 3(2-\eta)\lambda^2 + 3(2+\eta)\lambda + 4(1+\eta)]}{27\mu[3(\eta-1)\lambda^5 - (2+3\eta)]}. \tag{C 13}$$

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